Name: ______________________________

Date: ______________________________

Collaborators: ____

(If applicable, the collaborators submit their individually written assignments together)

Question:	1	2	Total
Points:	60	20	80
Score:			

Instructor/grader comments:

1. The second order non-linear autonomous differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \varepsilon\left(x^2 - 1\right)\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0, \quad \varepsilon > 0 \tag{1}$$

is called *van der Pol equation*. The equation models a non-conservative system in which energy is added to and subtracted from. The sign of the "coefficient" in the damping term in Eq. (1), $\left(x^2 - 1\right)$ changes, depending whether $|x|$ is larger or smaller than one, describing the inflow and outflow of the energy.

The equation was originally proposed in the 1920s to describe stable oscillations in electrical circuits employing vacuum tubes. By now, the van der Pol equation (under different names) has a long history of being used in physical and biological sciences.

Van der Pol oscillator is an example of a system that exibits the so called *limit cycle*. A limit cycle is an isolated closed trajectory $\dot{x} = \dot{x}(x)$ in the phase space $(x, \dot{x})$. Isolated means that neighboring trajectories are not closed; they spiral either toward or away from the limit cycle. If all neighboring trajectories approach the limit cycle, we say the limit cycle is stable or attracting. Otherwise the limit cycle is in general unstable.

Stable limit cycles model systems, e.g. the beating of a heart, that exhibit self-sustained oscillations. These systems oscillate even in the absence of external periodic forcing. There is a standard oscillation of some preferred period, waveform, and amplitude. If the system is perturbed slightly, it returns to the standard cycle.

Limit cycles are inherently nonlinear phenomena. They can't occur in linear systerns. Of course, a linear system, such as a linear differential equation, can have closed orbits – periodic solutions, but they won't be isolated. If $x(t)$ is a periodic solution, then so is $\alpha x(t)$ for any constant $\alpha \neq 0$. Hence $x(t)$ is surrounded by a 'family' of closed orbits. Consequently, the amplitude of a linear oscillation is set entirely by its initial conditions. Any slight disturbance to the amplitude will persist forever. In contrast, limit cycle oscillations are determined by the structure of the system itself.

Limit cycles are only possible in systems with dissipation. System that conserve energy do not have isolated closed trajectories.

(a) (10 points) On your virtual machine, **once per every project**: create a directory for your project (say `mkdir hw05`) and change to that directory; create an empty README.md file; download a sample .gitignore file and properly rename it (or copy it from one of your earlier projects). Start julia, activate the project and add packages you will use (`IJulia`, `PyPlot`, `OrdinaryDiffEq`).

Use jupyter notebook interface to write the code for this homework assignment. Place all code in a single notebook file (call it e.g. **hw05.ipynb**).

(b) (50 points) Consider the IVP Eq. (1) ~~with the initial conditions $x(0) = 1$, $\dot{x}(0) = 0$~~. Consider separately two cases: week nonlinearity, $\varepsilon = 0.1$, with the initial conditions $x(0) = 1.5$, $\dot{x}(0) = 0$, and strong nonlinearity, $\varepsilon = 10$, with the initial conditions $x(0) = 0.5$, $\dot{x}(0) = 0$. Solve Eq. (1) using Julia's `OrdinaryDiffEq` package. Consider the range of the independent variable t that ~~covers about five periods of limit cycles~~ is sufficiently long so that the trajectories settles on the limit cycles . (The ~~periods~~ ranges are different for week and strong nonlinearity cases.) On two different figures plot the phase trajectories $\dot{x}(x)$. (You must have sufficient number of data points so that the trajectories appear as smooth curves.) Provide the ~~legend,~~ grid, title, axes labels for each of your graphs.

Are the limit cycles of the van der Pol equation stable or unstable? Describe (in the README.md file of your git project) your reasoning as well as the qualitative difference between limit cycles for week and strong nonlinearity.

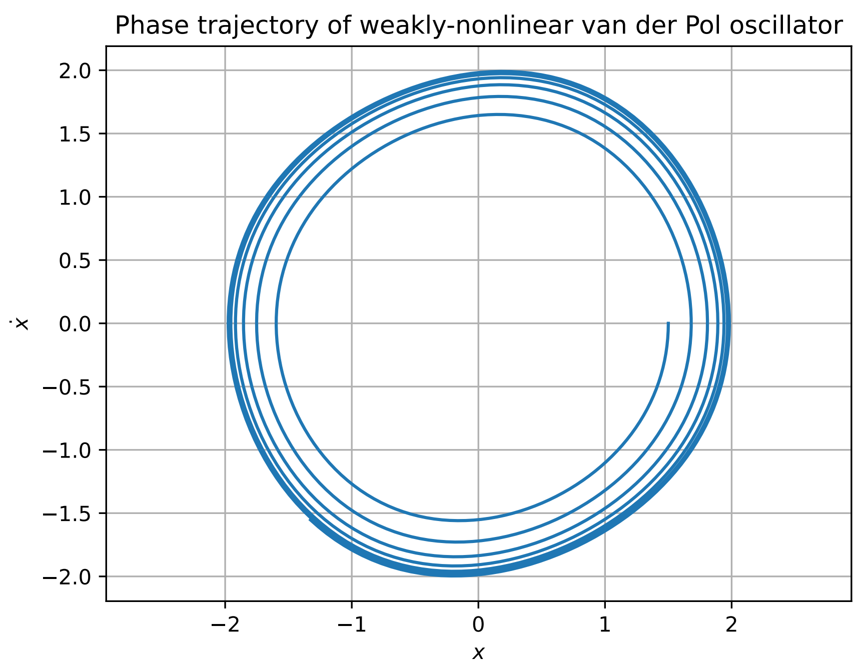

Figure 1:

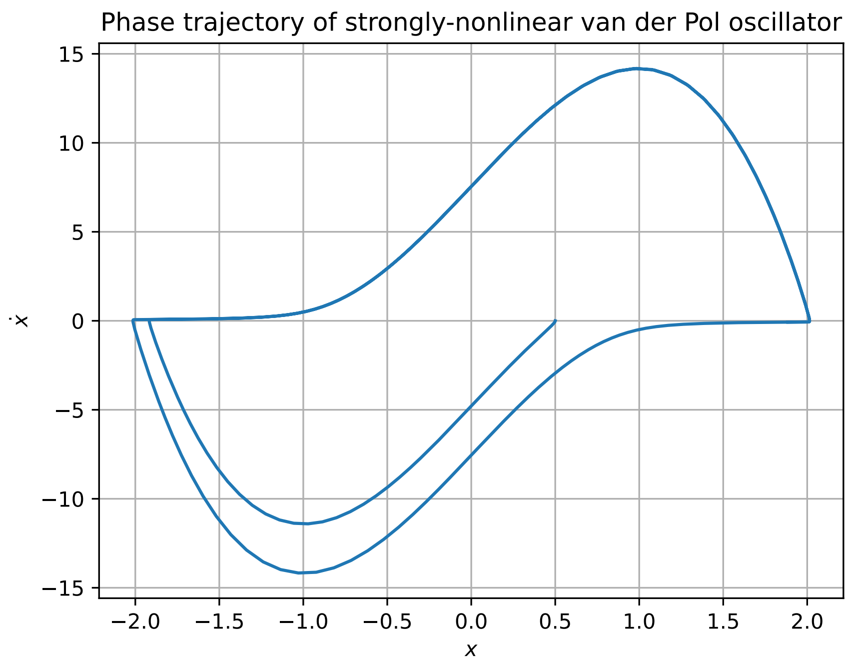

Figure 2:

2. (20 points)

   1. On the GitLab: Create an empty GitLab project called **hw05** (name it exactly as shown).
   2. On the VM: Clean the cells of your jupyter notebook and save the notebook. Delete unneeded notebooks if you created ones (e.g. `Untitled.ipynb`). Initialize a git repository for your project. Check in your notebook, Project.toml and Manifest.toml, an empty README.md file, and your .gitignore file into the repository. Provide a meaningful commit message. Push the content of your git repository to GitLab hw05 project.
   3. On the GitLab: Edit README.md file to add content as requested in Problem 1.
   4. On the VM: Pull the README.md file to your local git repository (`git pull`).
   5. On the GitLab: Share the project with the instructor (GitLab user name p2200_23f_in) and grant him **Reporter** privileges.