MONTE CARLO CALCULATIONS OF EULER-MASCHERONI CONSTANT

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The Euler–Mascheroni constant, $\gamma \approx 0.57\ldots$, is, next to π and e, one of the most important mathematical constants. It is defined as the limiting difference between the partial sum harmonic series and the natural logarithm of the number of terms in the series.

$$\gamma \equiv \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right). \tag{1}$$

An integral representation

One of many known integral representations of γ can be obtained as follows:

$$\begin{aligned}
\gamma &= \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln(n+1)\right) \\
&= \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \int_{1}^{n+1}\frac{\mathrm{d}x}{x}\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \sum_{k=1}^{n}\int_{k}^{k+1}\frac{\mathrm{d}x}{x}\right) \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{k} - \int_{k}^{k+1}\frac{\mathrm{d}x}{x}\right). \qquad (2)
\end{aligned}$$

Noting that

$$\frac{1}{k} = \int_{k}^{k+1}\frac{\mathrm{d}x}{\lfloor x\rfloor}, \tag{3}$$

where $\lfloor x \rfloor$ is the *floor function*, i.e. the function that takes as input a real number x, and gives as output the greatest integer less than or equal to x, we can rewrite Eq. (2) as follows:

$$\begin{aligned}\gamma &= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\int_k^{k+1}\frac{\mathrm{d}x}{\lfloor x\rfloor} - \int_k^{k+1}\frac{\mathrm{d}x}{x}\right)\\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\int_k^{k+1}\left(\frac{1}{\lfloor x\rfloor}-\frac{1}{x}\right)\mathrm{d}x = \lim_{n\to\infty}\int_1^{n+1}\left(\frac{1}{\lfloor x\rfloor}-\frac{1}{x}\right)\mathrm{d}x. \qquad (4)\end{aligned}$$

Next, note that

$$\int_1^{n+1}\frac{\lceil x\rceil}{x^2}\,\mathrm{d}x = \int_1^{n+1}\frac{1}{\lfloor x\rfloor}\,\mathrm{d}x. \qquad (5)$$

Here $\lceil x \rceil$ is the *ceiling function*, i.e. the function that takes as input a real number x, and gives as output the smallest integer greater than or equal to x.

$$\lceil x\rceil = 1 + \lfloor x\rfloor. \qquad (6)$$

Indeed,

$$\begin{aligned}\int_1^{n+1}\left(\frac{\lceil x\rceil}{x^2}-\frac{1}{\lfloor x\rfloor}\right)\mathrm{d}x &= \sum_{k=1}^{n}\int_k^{k+1}\left(\frac{\lceil x\rceil}{x^2}-\frac{1}{\lfloor x\rfloor}\right)\mathrm{d}x\\ &= \sum_{k=1}^{n}\int_k^{k+1}\left(\frac{k+1}{x^2}-\frac{1}{k}\right)\mathrm{d}x = \sum_{k=1}^{\infty}\left[-\frac{k+1}{x}-\frac{x}{k}\right]_{x=k}^{x=k+1}\\ &= \sum_{k=1}^{n}\left[-\frac{k+1}{k+1}-\frac{k+1}{k}+\frac{k+1}{k}+\frac{k}{k}\right] = \sum_{k=1}^{n}0 = 0. \qquad (7)\end{aligned}$$

Thus, we can rewrite the integral for γ Eq. (4) as follows:

$$\gamma = \lim_{n\to\infty}\int_1^{n+1}\left(\frac{\lceil x\rceil}{x^2}-\frac{1}{x}\right)\mathrm{d}x = \int_1^{\infty}\left(\frac{\lceil x\rceil}{x^2}-\frac{1}{x}\right)\mathrm{d}x. \qquad (8)$$

Monte Carlo algorithm

A Monte Carlo algorithm to calculate the Euler-Mascheroni constant by using uniform random variables and elementary functions, is as follows:

1. Generate uniformly distributed on $(0,1]$ random numbers u_i, $i = 1,\ldots,n$.
2. For every u_i calculate
$$w_i = 1 - \left\{\frac{1}{u_i}\right\}, \tag{9}$$
where $\{x\}$ denotes the fractional part of x,
$$\{x\} = x - \lfloor x \rfloor. \tag{10}$$
3. Calculate $\bar{w}(n)$ the average of w_i, $i = 1,\ldots,n$.
$$\bar{w}(n) \equiv \left(\frac{1}{n}\sum_{i=1}^{n} w_i\right). \tag{11}$$
4. The Euler-Mascheroni constant, γ, is
$$\gamma = \lim_{n\to\infty} \bar{w}(n) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\left(1 - \left\{\frac{1}{u_i}\right\}\right). \tag{12}$$

Indeed, for the unit uniform distribution,
$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\left(1 - \left\{\frac{1}{u_i}\right\}\right) = \int_0^1 \left(1 - \left\{\frac{1}{u}\right\}\right)\mathrm{d}u. \tag{13}$$

To evaluate the last integral, we introduce a new integration variable, x:
$$x = \frac{1}{u}, \quad \infty > x \geq 1, \quad u = \frac{1}{x}, \quad \mathrm{d}u = -\frac{\mathrm{d}x}{x^2}. \tag{14}$$

Then,
$$\begin{aligned} \lim_{n\to\infty} \bar{w}(n) &= -\int_\infty^1 (1 - \{x\})\frac{\mathrm{d}x}{x^2} = \int_1^\infty \frac{1 - (x - \lfloor x \rfloor)}{x^2}\mathrm{d}x \\ &= \int_1^\infty \left(\frac{1 + \lfloor x \rfloor}{x^2} - \frac{1}{x}\right)\mathrm{d}x = \int_1^\infty \left(\frac{\lceil x \rceil}{x^2} - \frac{1}{x}\right)\mathrm{d}x = \gamma. \end{aligned} \tag{15}$$

References

[1] Statistics Stack Exchange. *Estimate the Euler–Mascheroni constant by Monte Carlo simulations*. [Online; accessed 2023-11-24]. 2021. URL: https://stats.stackexchange.com/questions/531538/.