NEWTON-COTES QUADRATURE RULES

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https://www.phys.uconn.edu/~rozman/Courses/P2200_21F/

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1 Introduction

In numerical analysis, the Newton-Cotes formulas are a group of formulas for numerical integration based on evaluating the integrand at equally spaced points.

$$\int_a^b f(x)\,\mathrm{d}x \approx \sum_{i=1}^{n} w_i\, f(x_i), \quad x_{i+1} = x_i + h, \quad i = 2, \ldots, n. \tag{1}$$

h is called step size, w_i are called weights.

There are two classes of Newton-Cotes quadrature: they are called "closed" when $x_1 = a$ and $x_n = b$, i.e., they use the function values at the interval endpoints, and "open" when $x_1 > a$ and $x_n < b$, i.e., they do not use the function values at the endpoints.

One can (approximately) evaluate the integral over the interval $[x_i, x_{i+1}]$ by expanding the integrand into Taylor series about the midpoint, x_m:

$$x_m = \frac{x_{i+1} - x_i}{2} = x_i + \frac{h}{2} = x_{i+1} - \frac{h}{2}. \tag{2}$$

$$\begin{aligned}
\int_{x_i}^{x_{i+1}} f(x)\,\mathrm{d}x &\approx \int_{x_i}^{x_{i+1}} \left\{ f(x_m) + f'(x_m)(x-x_m) + \frac{1}{2} f''(x_m)(x-x_m)^2 \right\} \mathrm{d}x \qquad (3)\\
&= f(x_m) \int_{x_i}^{x_{i+1}} \mathrm{d}x + f'(x_m) \int_{x_i}^{x_{i+1}} (x-x_m)\,\mathrm{d}x + \frac{1}{2} f''(x_m) \int_{x_i}^{x_{i+1}} (x-x_m)^2\,\mathrm{d}x \qquad (4)\\
&= f(x_m)h + f'(x_m) \int_{-h/2}^{h/2} u\,\mathrm{d}u + \frac{1}{2} f''(x_m) \int_{-h/2}^{h/2} u^2\,\mathrm{d}u \qquad (5)\\
&= h f(x_m) + \frac{h^3}{24} f''(x_m) + O(h^5) \qquad (6)
\end{aligned}$$

2 Midpoint quadrature

$$\int_{x_i}^{x_{i+1}} f(x)\,\mathrm{d}x \approx \int_{x_i}^{x_{i+1}} f(x_m)\,\mathrm{d}x = M = h f(x_m). \qquad (7)$$

Comparing Eq. (6) and Eq. (7), we conclude that the leading error of midpoint quadrature is as follows:

$$M = \int_{x_i}^{x_{i+1}} f(x)\,\mathrm{d}x - \frac{h^3}{24} f''(x_m) + O(h^5). \qquad (8)$$

3 Trapezoidal quadrature

$$\int_{x_i}^{x_{i+1}} f(x)\,\mathrm{d}x \approx \frac{h}{2} \left(f(x_i) + f(x_{i+1}) \right) = T. \qquad (9)$$

Approximating $f(x_i)$ and $f(x_{i+1})$ by their Taylor expansion about x_m:

$$\begin{aligned}
f(x_i) &= f(x_m) - \frac{h}{2} f'(x_m) + \frac{1}{2} \left(\frac{h}{2} \right)^2 f''(x_m), \qquad (10)\\
f(x_{i+1}) &= f(x_m) + \frac{h}{2} f'(x_m) + \frac{1}{2} \left(\frac{h}{2} \right)^2 f''(x_m), \qquad (11)
\end{aligned}$$

we obtain

$$T = f(x_m)h + \frac{h^3}{8} f''(x_m). \qquad (12)$$

Comparing Eq. (6) and Eq. (12), we conclude that

$$T = \int_{x_i}^{x_{i+1}} f(x)\,\mathrm{d}x + \frac{h^3}{12} f''(x_m) + O(h^5). \tag{13}$$

4 Simpson's quadrature

We can combine Eq. (8) and Eq. (13) such that the error term of order h^3 cancels out:

$$\frac{1}{3}(2M + T) = \int_{x_i}^{x_{i+1}} f(x)\,\mathrm{d}x + O(h^5). \tag{14}$$

The new quadrature rule is called Simpson's quadrature:

$$\int_{x_i}^{x_{i+1}} f(x)\,\mathrm{d}x \approx \frac{h}{6}\left(f(x_i) + 4f(x_m) + f(x_{i+1})\right). \tag{15}$$

5 Composite Simpson's rule

$$\int_a^b f(x)\,\mathrm{d}x = \int_{x_1}^{x_2} f(x)\,\mathrm{d}x + \int_{x_2}^{x_3} f(x)\,\mathrm{d}x + \ldots \tag{16}$$

$$= \frac{h}{3}\left(f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + \ldots + 4f(x_{n-1}) + f(x_n)\right) + O(h^4), \tag{17}$$

where

$$h = \frac{b-a}{n-1}. \tag{18}$$