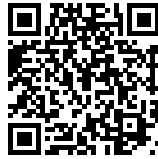


RECURRENCE RELATIONS

http://www.phys.uconn.edu/~rozman/Courses/m3510_17f/



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Consider the following integral:

$$I_n = \int_0^1 x^n e^{-x} dx, \quad (1)$$

where n is non-negative integer, $n = 0, 1, 2, \dots$

Integral Eq. (1) is related to incomplete gamma function:

$$\gamma(s, t) \equiv \int_0^t x^{s-1} e^{-x} dx. \quad (2)$$

Namely,

$$I_n \equiv \gamma(n+1, 1). \quad (3)$$

From Eq. (1), all I_n are positive, $I_n > 0$. Moreover, for large n , $n \gg 1$,

$$I_n \approx e^{-1} \int_0^1 x^n dx = \frac{e^{-1}}{n+1}, \quad (4)$$

i.e.

$$\lim_{n \rightarrow \infty} I_n = 0. \quad (5)$$

Integrating by parts we find the following relation:

$$I_n = \int_0^1 x^n \mathrm{d}(-e^{-x}) = -x^n e^{-x} \Big|_0^1 + n \int_0^1 x^{n-1} e^{-x} \mathrm{d}x = -\frac{1}{e} + nI_{n-1}, \quad (6)$$

$$\boxed{I_n = nI_{n-1} - \frac{1}{e}}. \quad (7)$$

To be able to use Eq. (7) we need to supplement Eq. (7) by an initial condition:

$$I_0 = \int_0^1 e^{-x} \mathrm{d}x = 1 - \frac{1}{e}. \quad (8)$$