RECURRENCE RELATIONS

http://www.phys.uconn.edu/~rozman/Courses/m3510_17f/

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Consider the following integral:

$$I_n = \int_0^1 x^n e^{-x} \mathrm{d}x, \tag{1}$$

where n is non-negative integer, $n = 0, 1, 2, \ldots$

Integral Eq. (1) is related to incomplete gamma function:

$$\gamma(s,t) \equiv \int_0^t x^{s-1} e^{-x} \mathrm{d}x. \tag{2}$$

Namely,

$$I_n \equiv \gamma(n+1, 1). \tag{3}$$

From Eq. (1), all I_n are positive, $I_n > 0$. Moreover, for large n, $n \gg 1$,

$$I_n \approx e^{-1} \int_0^1 x^n \mathrm{d}x = \frac{e^{-1}}{n+1}, \tag{4}$$

i.e.

$$\lim_{n\to\infty} I_n = 0. \tag{5}$$

Integrating by parts we find the following relation:

$$I_n = \int\limits_0^1 x^n \mathrm{d}\left(-e^{-x}\right) = -\, x^n e^{-x}\big|_0^1 + n \int\limits_0^1 x^{n-1} e^{-x} \mathrm{d}x = -\frac{1}{e} + n I_{n-1}, \tag{6}$$

$$\boxed{I_n = n\, I_{n-1} - \frac{1}{e}}. \tag{7}$$

To be able to use Eq. (7) we need to supplement Eq. (7) by an initial condition:

$$I_0 = \int_0^1 e^{-x} \mathrm{d}x = 1 - \frac{1}{e}. \tag{8}$$