Geometric calculation of π

Last modified: October 6, 2015

Consider a regular n-sided polygons inscribed in a circle of radius r.

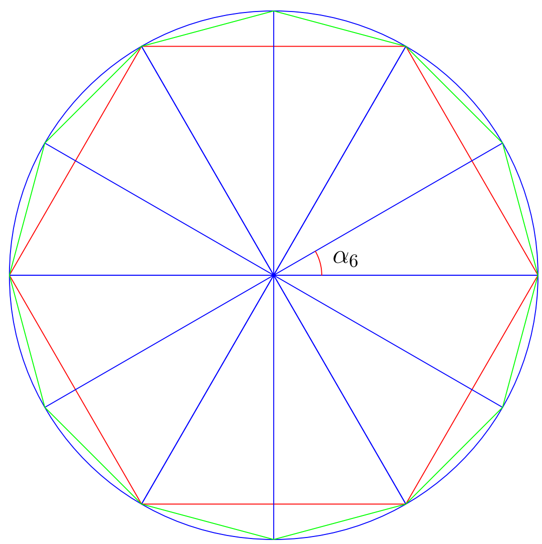

Figure 1: Regular polygon inscribed in a circle

Let l_n denotes the perimeter of the polygon. Consider the following ratio, p_n,

$$p_n \equiv \frac{l_n}{2r} \tag{1}$$

In the limit $n \to \infty$, a sequence of regular polygons with an increasing number of sides becomes a circle, therefore

$$\lim_{n\to\infty} l_n = 2\pi r \tag{2}$$

and

$$\lim_{n\to\infty} p_n = \pi. \tag{3}$$

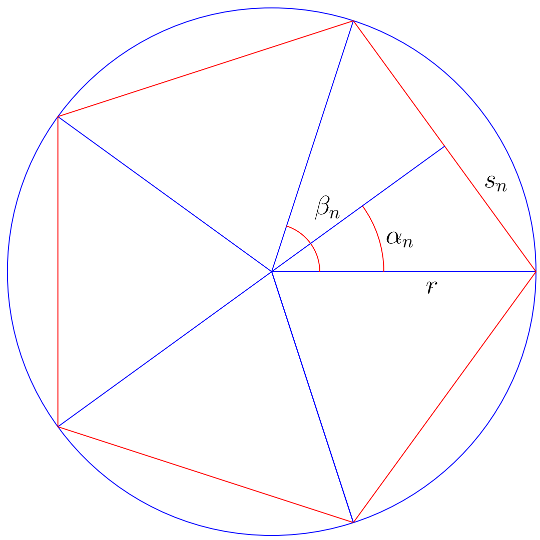

Figure 2: The length of a half-side of a regular polygon, $s_n = r \sin \alpha_n$, where $\alpha_n = \frac{\beta_n}{2}$, $\beta_n = \frac{2\pi}{n}$

On the other hand,

$$l_n = 2 s_n n = 2nr \sin \alpha_n, \tag{4}$$

where s_n is the length of a half of the polygon side (See Fig 2). Therefore,

$$p_n = n \sin \alpha_n, \tag{5}$$

where

$$\alpha_n = \frac{1}{2}\left(\frac{2\pi}{n}\right) = \frac{\pi}{n}. \tag{6}$$

Let's double the number of the polygon's sides:

$$p_{2n} = 2n \sin \alpha_{2n} \tag{7}$$

$$\alpha_{2n} = \frac{\pi}{2n} = \frac{\alpha_n}{2}, \qquad \text{or} \quad \alpha_n = 2\alpha_{2n}. \tag{8}$$

$$\sin \alpha_n = \sin(2\alpha_{2n}) = 2 \sin \alpha_{2n} \cos \alpha_{2n} \tag{9}$$

$$= 2 \sin \alpha_{2n} \sqrt{1 - \sin^2 \alpha_{2n}}. \tag{10}$$

Let's introduce the notation γ_n:

$$\gamma_n \equiv \sin\alpha_n, \quad \gamma_{2n} \equiv \sin\alpha_{2n}. \tag{11}$$

$$\gamma_n = 2\gamma_{2n}\sqrt{1-\gamma_{2n}^2} \tag{12}$$

$$\gamma_n^2 = 4\gamma_{2n}^2\left(1-\gamma_{2n}^2\right) \tag{13}$$

$$\gamma_{2n}^4 - \gamma_{2n}^2 + \frac{1}{4}\gamma_n^2 = 0 \tag{14}$$

$$\gamma_{2n}^2 = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{4}\gamma_n^2} = \frac{1-\sqrt{1-\gamma_n^2}}{2} \tag{15}$$

$$\sin\alpha_{2n} = \sqrt{\frac{1-\sqrt{1-\sin^2\alpha_n}}{2}} \tag{16}$$

We supplement the recurrence relation Eq. (refeq:14) by the initial condition

$$\sin\alpha_8 = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}. \tag{17}$$

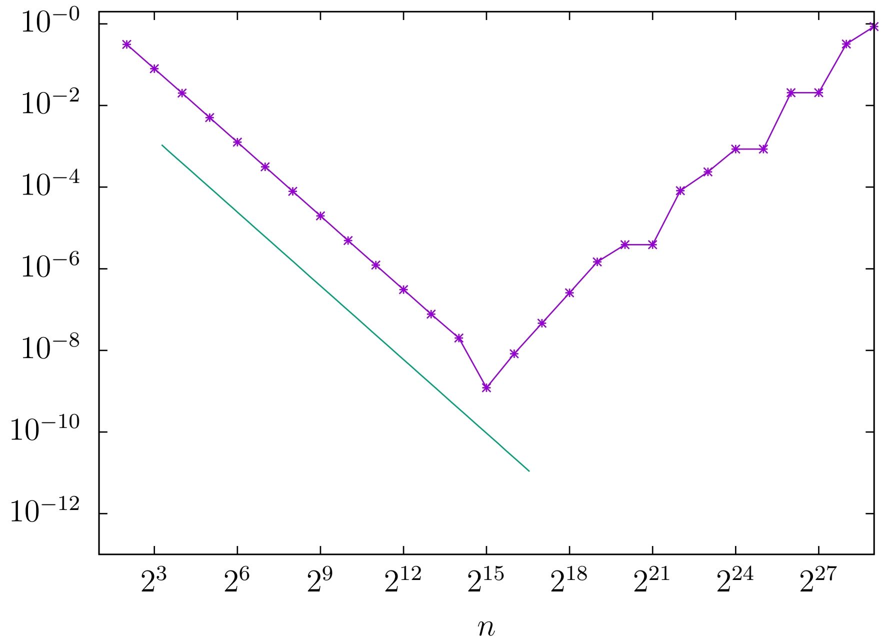

Figure 3: Absolute error in calculating π via polygon geometry. Graph of $\frac{1}{x^2}$ is provided for comparison.

```
set terminal epslatex input
set output 'pi-poly-gp.tex'

set logscale x 2
set format x ""
set xtics ( "$2^3$" 8., "$2^6$" 64., "$2^9$" 512., "$2^{12}$" 4096., \
           "$2^{15}$" 32768., "$2^{18}$" 262144., "$2^{21}$" 2097152., \
           "$2^{24}$" 16777216., "$2^{27}$" 134217728.)

set logscale y 10
set format y ""
set ytics ( "$10^{-14}$" 1e-14, "$10^{-12}$" 1e-12, "$10^{-10}$" 1e-10, "$10^{-8}$" 1.e-8, "$10^{-6}$" 1e-6, \
            "$10^{-4}$" 1.e-4, "$10^{-2}$" 1e-2, "$10^{-0}$" 1)

set mxtics
set mytics

#set title 'Absolute error in calculating $\pi$ via polygon geometry'
set xlabel '$n$'
#set ylabel '$\left| p_n - \pi \right|$' offset 5,0


unset key

#p [2:] [1e-13:2] "pi-poly-u.res" u ($2):(abs($5)) w lp pt 3, \
#               "pi-poly-s.res" u ($2):(abs($5)) w lp pt 4, \
#              (x>8?(x<100000.?0.1/(x*x):1/0):1/0) w l

p [2:] [1e-13:2] "pi-poly-u.res" u ($2):(abs($5)) w lp pt 3, \
             (x>8?(x<100000.?0.1/(x*x):1/0):1/0) w l
```

Figure 4: Gnuplot script that produced Figure 3.