The Three-Body Problem

Adapted from Richard Fitzpatrick, *Newtonian Dynamics*

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1 Introduction

An isolated dynamical system consisting of two moving point masses exerting forces on one another — which is usually referred to as a *two-body problem* — can always be converted into an equivalent one-body problem. In particular, this implies that we can *exactly solve* a dynamical system containing *two* gravitationally interacting point masses, since the equivalent one-body problem is exactly soluble. What about a system containing *three* gravitationally interacting point masses? Despite hundreds of years of research, no exact solution of this famous problem — which is generally known as the *three-body problem* — has ever been found. It is, however, possible to make some progress by restricting the problem's scope.

2 The Circular Restricted Three-Body Problem

Consider a mechanical system consisting of three gravitationally interacting point masses, M_1, M_2, and m. Suppose, that the third mass, m, is so much smaller than the other two that it has a negligible effect on their motion. Suppose, further, that the first two masses, M_1 and M_2, execute a circular orbits about their common center of mass. This simplified problem is known as the circular restricted three-body problem.

Let us further assume, to simplify the presentation of the final calculations, that mass m moves in the plane of the orbital motion of masses M_1 and M_2.

Let ω be the constant orbital angular velocity of masses M_1 and M_2 on the circular orbit. We can find ω by equating F_{cp}, the centripetal force acting upon the mass $\mu = \frac{M_1 M_2}{M_1 + M_2}$ (the equivalent one-body problem), and F_g, the force of gravitational attraction between masses M_1 and M_2:

$$F_{cp} = \frac{M_1 M_2}{M_1 + M_2}\,\frac{v^2}{R}, \qquad F_g = G\,\frac{M_1 M_2}{R^2}, \tag{1}$$

where G is the gravitational constant, v is the constant linear velocity of mass μ. From Eq. (1)

$$v^2 = G\,\frac{M_1 + M_2}{R}. \tag{2}$$

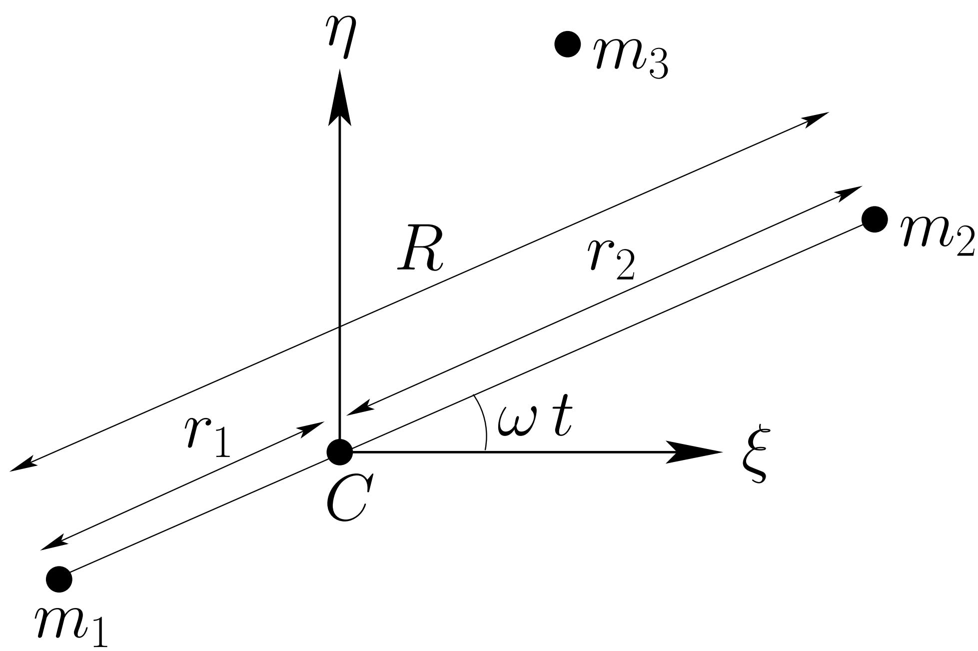

Figure 1: The circular restricted three-body problem.

On the other hand, the period of orbital motion on a circular orbit, T, is

$$T = \frac{2\pi R}{v}, \tag{3}$$

thus,

$$\omega \equiv \frac{2\pi}{T} = \frac{v}{R}. \qquad \omega^2 = \frac{v^2}{R^2}. \tag{4}$$

Substituting Eq. (2) into Eq. (4), we arrive at the following expression.

$$\omega^2 = G\,\frac{M_1 + M_2}{R^3}. \tag{5}$$

Let us define a Cartesian coordinate system $(\xi,\, \eta,\, \zeta)$ in an inertial reference frame whose origin coincides with the center of mass, C, of the two orbiting masses, M_1 and M_2. Let the orbital plane of these masses coincide with the ξ-η plane, and let them both lie on the ξ-axis at time $t = 0$ — see Figure 1. Suppose that R is the constant distance between the two orbiting masses, r_1 the constant distance between mass M_1 and the origin, and r_2 the constant distance between mass M_2 and the origin.

Let the third mass have position vector $\vec{r} = (\xi,\, \eta,\, 0)$. The Cartesian components of the equation of motion of this mass are thus

$$\ddot{\xi} \;=\; -GM_1\,\frac{(\xi - \xi_1)}{\rho_1^3} - GM_2\,\frac{(\xi - \xi_2)}{\rho_2^3}, \tag{6}$$

$$\ddot{\eta} \;=\; -GM_1\,\frac{(\eta - \eta_1)}{\rho_1^3} - GM_2\,\frac{(\eta - \eta_2)}{\rho_2^3}, \tag{7}$$

where

$$\begin{aligned} \rho_1^2 &= (\xi - \xi_1)^2 + (\eta - \eta_1)^2, &(8) \\ \rho_2^2 &= (\xi - \xi_2)^2 + (\eta - \eta_2)^2. &(9) \end{aligned}$$

3 Co-Rotating Frame

Let us transform to a non-inertial frame of reference rotating with angular velocity ω about an axis normal to the orbital plane of masses M_1 and M_2, and passing through their center of mass. The masses M_1 and M_2 are *stationary* in this new reference frame. Let us define a Cartesian coordinate system $(X,\, Y)$ in the rotating frame of reference which is such that masses M_1 and M_2 always lie on the X-axis. Let the position vector of mass m be $\vec{r} = (x,\, y)$ — see Figure 2.

The masses M_1 and M_2 have the fixed position vectors

$$\vec{r}_1 = (-\alpha R,\, 0,\, 0) \qquad \vec{r}_2 = ((1-\alpha)R,\, 0,\, 0) \tag{10}$$

in our new coordinate system. Indeed, by the definition of the center of mass,

$$r_1 M_1 = r_2 M_2. \tag{11}$$

on the other hand,

$$r_1 + r_2 = R. \tag{12}$$

Solving Eqs. (11) and (12), we obtain,

$$r_1 = \frac{M_2}{M_1 + M_2}\, R, \qquad r_2 = \frac{M_1}{M_1 + M_2}\, R = \left(1 - \frac{M_2}{M_1 + M_2}\right) R, \tag{13}$$

i.e. in Eq. (10)

$$\alpha = \frac{M_2}{M_1 + M_2} \tag{14}$$

The equation of motion of mass m in the rotating reference frame are obtained by including into Eqs. (6), (7) two additional forces — Coriolis force $\vec{F}_{cor}$ and centrifugal force $\vec{F}_{cf}$:

$$\vec{F}_{cf} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = m\omega^2 \vec{r}, \tag{15}$$

$$\vec{F}_{cor} = -2m\vec{\omega} \times \dot{\vec{r}} = 2m\omega\,(-\hat{x}\dot{y} + \hat{y}\dot{x}) \tag{16}$$

$$\ddot{\vec{r}} = -GM_1\,\frac{(\mathbf{r} - \mathbf{r}_1)}{\rho_1^3} - GM_2\,\frac{(\vec{r} - \vec{r}_2)}{\rho_2^3} - \vec{\omega} \times (\vec{\omega} \times \mathbf{r}) - 2\,\vec{\omega} \times \dot{\mathbf{r}}, \tag{17}$$

where $\vec{\omega} = (0,\, 0,\, \omega)$, and

$$\begin{aligned} \rho_1^2 &= (x + \alpha R)^2 + y^2, &(18) \\ \rho_2^2 &= (x - (1-\alpha)R)^2 + y^2. &(19) \end{aligned}$$

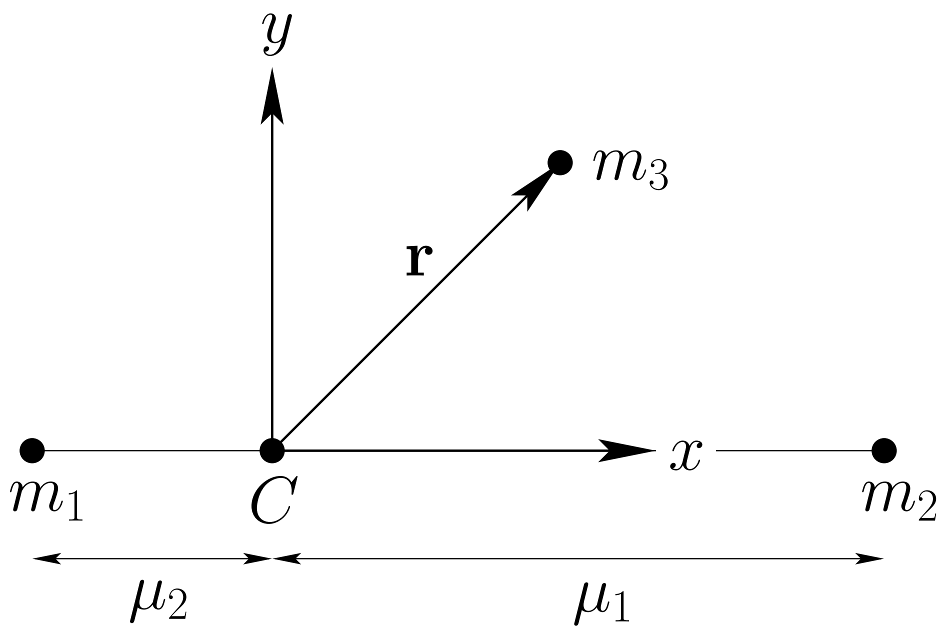

Figure 2: The co-rotating frame.

Here, the last two terms on the right-hand side of Eq. (17) are the *centrifugal* acceleration and the *Coriolis* acceleration.

The components of Eq. (17) reduce to

$$\ddot{x} = -\frac{GM_1\,(x+\alpha R)}{\rho_1^3} - \frac{GM_2\,(x-(1-\alpha)R)}{\rho_2^3} + \omega^2\,x + 2\,\omega\,\dot{y}, \tag{20}$$

$$\ddot{y} = -\frac{GM_1\,y}{\rho_1^3} - \frac{GM_2\,y}{\rho_2^3} + \omega^2\,y - 2\,\omega\,\dot{x}. \tag{21}$$

4 Jacobi integral

Eqs. (20), (21) can be rewritten as following.

$$\ddot{x} - 2\,\omega\,\dot{y} = -\frac{\partial U}{\partial x}, \tag{22}$$

$$\ddot{y} + 2\,\omega\,\dot{x} = -\frac{\partial U}{\partial y}. \tag{23}$$

where

$$U = -\frac{GM_1}{\rho_1} - \frac{GM_2}{\rho_2} - \frac{\omega^2}{2}\,(x^2+y^2) \tag{24}$$

is the sum of the gravitational and centrifugal potentials.

Now, it follows from Eqs (22)–(23) that

$$\ddot{x}\,\dot{x} - 2\,\omega\,\dot{x}\,\dot{y} = -\dot{x}\,\frac{\partial U}{\partial x}, \tag{25}$$

$$\ddot{y}\,\dot{y} + 2\,\omega\,\dot{x}\,\dot{y} = -\dot{y}\,\frac{\partial U}{\partial y}. \tag{26}$$

Summing the above equations, we obtain

$$\frac{d}{dt}\left[\frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + U\right] = 0. \tag{27}$$

In other words,

$$C = -2\,U - v^2 \tag{28}$$

is a *constant of the motion*, where $v^2 = \dot{x}^2 + \dot{y}^2$. C is called the *Jacobi integral*. The mass m is restricted to regions in which

$$-2\,U \geq C, \tag{29}$$

since v^2 is a positive definite quantity.

5 Dimensionless form of the equations

No analytic solutions of Eqs. (20)– (21) are known. Our goal is to solve them numerically. As the first required step, we convert the to a dimensionless form.

Circular restricted three body problem has two natural scales: the distance, R, between masses M_1 and M_2, and the characteristic time of their orbital motion $1/\omega$. Let us introduce dimensionless variables by measuring the coordinates x and y in units of R, thus introducing new unknowns u and v as following,

$$u \equiv \frac{x}{R}, \qquad v \equiv \frac{y}{R}, \tag{30}$$

Let us measure time t in units of $1/\omega$, introducing dimensionless variable τ,

$$\tau \equiv \omega t. \tag{31}$$

"Old" derivatives with respect to time are going to have the following forms:

$$\dot{x} \equiv \frac{dx}{dt} = \frac{d(u\,R)}{d(\tau/\omega)} = \omega R\,\frac{du}{d\tau}, \tag{32}$$

$$\ddot{x} \equiv \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}\left(\omega R\,\frac{du}{d\tau}\right) = \omega R\,\frac{d}{d\tau/\omega}\left(\frac{du}{d\tau}\right) = \omega^2 R\,\frac{d^2u}{d\tau^2}. \tag{33}$$

Similarly,

$$\dot{y} = \omega R\,\frac{dv}{d\tau} \tag{34}$$

$$\ddot{y} = \omega^2 R \, \frac{d^2 v}{d\tau^2} \tag{35}$$

Substituting Eqs. (32)– (35) into Eqs. (20), (21), we get:

$$\omega^2 R \, \frac{d^2 u}{d\tau^2} = -\frac{G M_1 R \, (u+\alpha)}{\rho_1^3} - \frac{G M_2 R \, (u-1+\alpha)}{\rho_2^3} + \omega^2 R \, u + 2 \, \omega^2 R \, \frac{dv}{d\tau}, \tag{36}$$

$$\omega^2 R \, \frac{d^2 v}{d\tau^2} = -\frac{G M_1 R \, v}{\rho_1^3} - \frac{G M_2 R \, v}{\rho_2^3} + \omega^2 R \, v - 2 \, \omega^2 R \, \frac{du}{d\tau}. \tag{37}$$

Here ρ_1 and ρ_2 expressed via dimensionless parameters are as following:

$$\rho_1 = R \left((u+\alpha)^2 + v^2 \right)^{\frac{1}{2}} = R d_1, \tag{38}$$

$$\rho_2 = R \left((u-1+\alpha)^2 + v^2 \right)^{\frac{1}{2}} = R d_2, \tag{39}$$

where

$$d_1 \equiv \left((u+\alpha)^2 + v^2 \right)^{\frac{1}{2}}, \tag{40}$$

$$d_2 \equiv \left((u-1+\alpha)^2 + v^2 \right)^{\frac{1}{2}}. \tag{41}$$

Dividing each term in Eqs. (36)– (37) by $\omega^2 R$, we arrive at the following equations:

$$\frac{d^2 u}{d\tau^2} = -\frac{G M_1}{\omega^2 R^3} \, \frac{(u+\alpha)}{d_1^3} - \frac{G M_2}{\omega^2 R^3} \, \frac{(u-1+\alpha)}{d_2^3} + u + 2 \, \frac{dv}{d\tau}, \tag{42}$$

$$\frac{d^2 v}{d\tau^2} = -\frac{G M_1}{\omega^2 R^3} \, \frac{v}{d_1^3} - \frac{G M_2}{\omega^2 R^3} \, \frac{v}{d_2^3} + v - 2 \, \frac{du}{d\tau}. \tag{43}$$

Noticing that

$$\frac{G M_1}{\omega^2 R^3} = \frac{M_1}{M_1 + M_2} \equiv 1 - \alpha \tag{44}$$

and

$$\frac{G M_2}{\omega^2 R^3} = \frac{M_2}{M_1 + M_2} \equiv \alpha \tag{45}$$

we arrive at the following equations.

$$\frac{d^2 u}{d\tau^2} = -(1-\alpha) \, \frac{(u+\alpha)}{d_1^3} - \alpha \, \frac{(u-1+\alpha)}{d_2^3} + u + 2 \, \frac{dv}{d\tau}, \tag{46}$$

$$\frac{d^2 v}{d\tau^2} = -(1-\alpha) \, \frac{v}{d_1^3} - \alpha \, \frac{v}{d_2^3} + v - 2 \, \frac{du}{d\tau}. \tag{47}$$

Equations (46)- (47) can be rewritten in a compact form

$$\ddot{u} = -\frac{\partial U}{\partial v} + 2 \, \dot{v}, \tag{48}$$

$$\ddot{v} = -\frac{\partial U}{\partial v} - 2 \, \dot{u}, \tag{49}$$

where

$$U(u,v) = -\frac{1-\alpha}{d_1} - \frac{\alpha}{d_2} - \frac{1}{2}\left(u^2+v^2\right) \tag{50}$$

is the dimensionless version of Eq. (24).

Equations (46)- (47) are dimensionless and contain a single parameter, α. Some of the results of their numerical solution are presented in Figs. 3 and 4. A fragment of the code used for calculations is presented in the Appendix A.

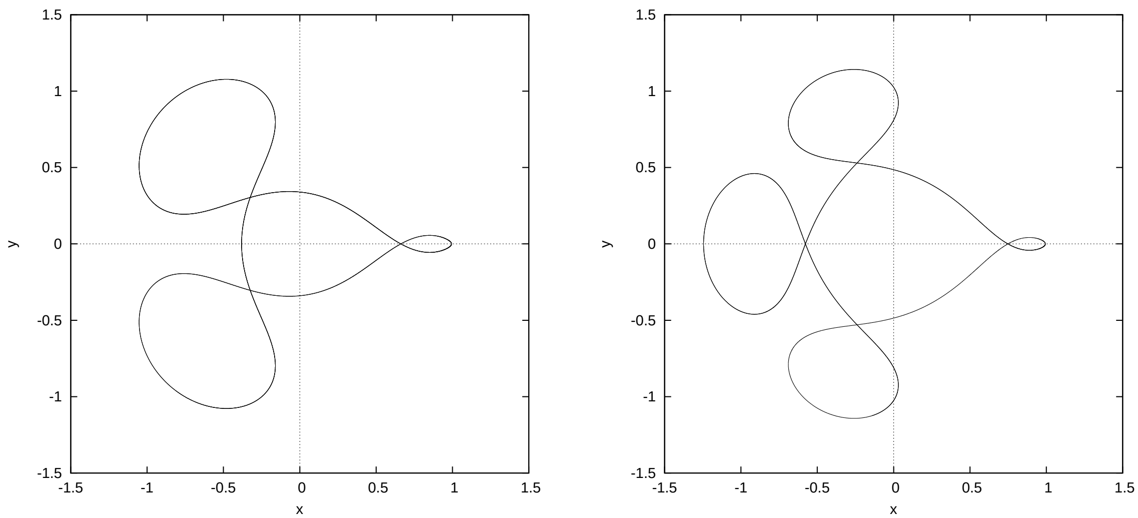

Figure 3: Arenstorf periodic orbits for $\alpha = 0.012277471$ and initial conditions $x(0) = 0.994$, $y(0) = 0$, $\dot{x}(0) = 0$; left subfigure: $\dot{y}(0) = -2.0317326295573368357302057924$, right subfigure: $\dot{y}(0) = -2.0015851063790825224053786224$,

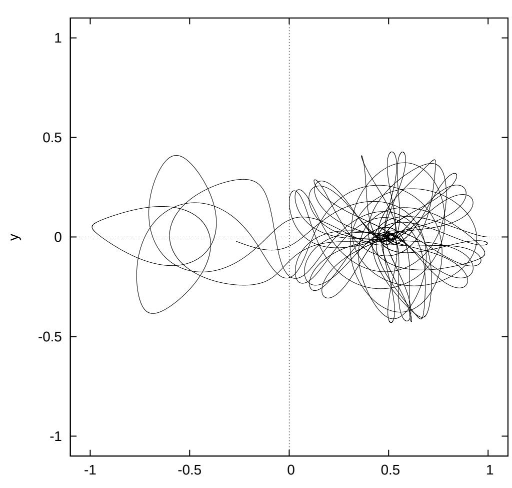

Figure 4: Chaotic orbit: $\alpha = 0.5$, $x(0) = 1$, $y(0) = 0$, $\dot{x}(0) = 0$, $\dot{y}(0) = 0$.

Appendix A

A fragment of a C code to solve the restricted three-body problem using gsl library.

```
int func (double t, const double yy[], double f[], void *params)
{
  double a = *(double *) params;
  double d1, d2;
  double x = yy[0], y = yy[1], vx = yy[2], vy = yy[3];

  d1 = pow((x + a)*(x + a) + y*y, 1.5);
  d2 = pow((x + a - 1.)*(x + a - 1.) + y*y, 1.5);

  f[0] = vx;
  f[1] = vy;
  f[2] = -(1. - a)*(x + a)/d1 - a*(x + a - 1.)/d2 + x + 2*vy,
  f[3] = -(1. - a)*y/d1 - a*y/d2 + y - 2*vx;

  return GSL_SUCCESS;
}
```