Geometric calculation of π

Last modified: September 17, 2013

Consider a regular n-sided polygons inscribed in a circle of radius r.

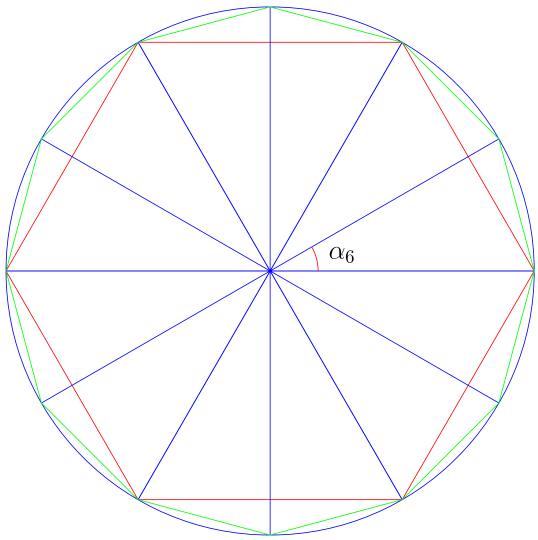

Figure 1: Regular polygon inscribed in a circle

Let l_n denotes the perimeter of the polygon. Consider the following ratio, p_n,

$$p_n \equiv \frac{l_n}{2r} \tag{1}$$

In the limit $n \to \infty$, a sequence of regular polygons with an increasing number of sides becomes a circle, therefore

$$\lim_{n\to\infty} l_n = 2\pi r \tag{2}$$

and

$$\lim_{n\to\infty} p_n = \pi. \tag{3}$$

On the other hand,

$$l_n = n(2S_n) = 2nr \sin\alpha_n. \tag{4}$$

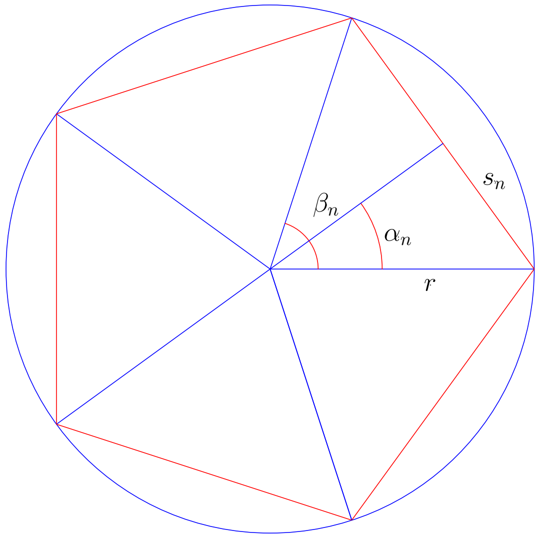

Figure 2: Half-sede of a regular polygon, $s_n = r \sin \alpha_n$, where $\alpha_n = \frac{\beta_n}{2}$, $\beta_n = \frac{2\pi}{n}$

(See Fig 2). Therefore,

$$p_n = n \sin \alpha_n, \tag{5}$$

where

$$\alpha_n = \frac{1}{2}\left(\frac{2\pi}{n}\right) = \frac{\pi}{n}. \tag{6}$$

Let's double the number of the poligon's sides:

$$p_{2n} = 2n \sin \alpha_{2n} \tag{7}$$

$$\alpha_{2n} = \frac{\pi}{2n} = \frac{\alpha_n}{2}, \qquad \text{or} \quad \alpha_n = 2\alpha_{2n}. \tag{8}$$

$$\sin \alpha_n = \sin (2\alpha_{2n}) = 2 \sin \alpha_{2n} \cos \alpha_{2n} \tag{9}$$

$$= 2 \sin \alpha_{2n} \sqrt{1 - \sin^2 \alpha_{2n}}. \tag{10}$$

Let's introduce the notation s_n:

$$s_n \equiv \sin \alpha_n, \quad s_{2n} \equiv \sin \alpha_{2n}. \tag{11}$$

$$s_n = 2s_{2n}\sqrt{1 - s_{2n}^2} \tag{12}$$

$$s_n^2 = 4s_{2n}^2\left(1 - s_{2n}^2\right) \tag{13}$$

$$s_{2n}^4 - s_{2n}^2 + \frac{1}{4}s_n^2 = 0 \tag{14}$$

$$s_{2n}^2 = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{4}s_n^2} = \frac{1-\sqrt{1-s_n^2}}{2} \tag{15}$$

$$\sin\alpha_{2n} = \sqrt{\frac{1-\sqrt{1-\sin^2\alpha_n}}{2}} \tag{16}$$

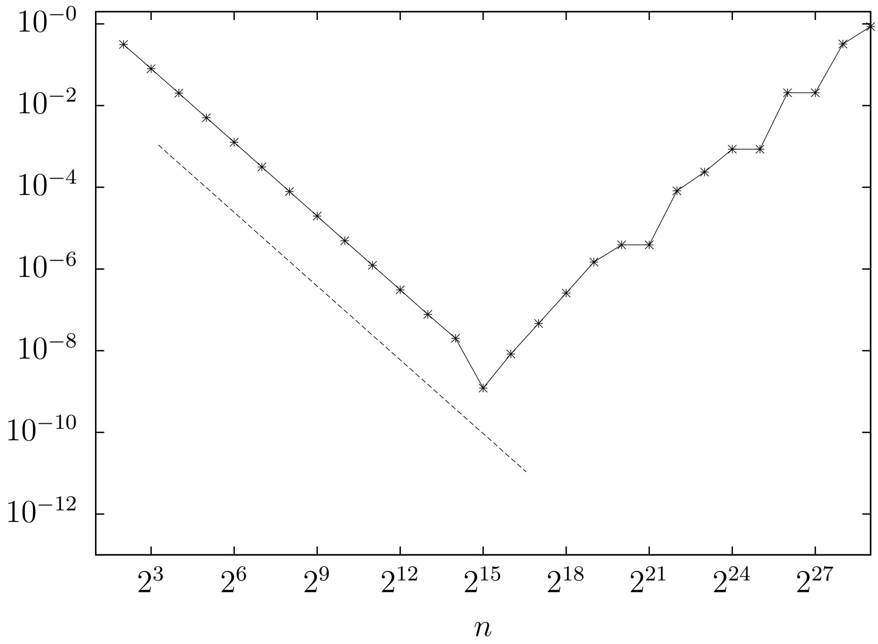

Figure 3: Absolute error in calculating π via polygon geometry. Graph of $\frac{1}{x^2}$ is provided for comparison.

```
set terminal epslatex input
set output 'pi-poly-gp.tex'

set logscale x 2
set format x ""
set xtics ( "$2^3$" 8., "$2^6$" 64., "$2^9$" 512., \
            "$2^{12}$" 4096., "$2^{15}$" 32768., \
            "$2^{18}$" 262144., "$2^{21}$" 2097152., \
            "$2^{24}$" 16777216., "$2^{27}$" 134217728.)

set logscale y 10
set format y ""
set ytics ( "$10^{-14}$" 1e-14, "$10^{-12}$" 1e-12,
            "$10^{-10}$" 1e-10, "$10^{-8}$" 1.e-8, \
            "$10^{-6}$" 1e-6, "$10^{-4}$" 1.e-4, \
            "$10^{-2}$" 1e-2, "$10^{-0}$" 1)

set mxtics
set mytics

set xlabel '$n$'

unset key

p [2:] [1e-13:2] "res.unstable" u ($2):(abs($5)) w lp pt 3, \
             (x>8?(x<100000.?0.1/(x*x):1/0):1/0) w l
```

Figure 4: Gnuplot script that produced Figure 3.