

INTRODUCTION TO FOURIER TRANSFORMS FOR PHYSICISTS

JAMES G. O'BRIEN

As we will see in the next section, the Fourier transform is developed from the Fourier integral, so it shares many properties of the former. First and foremost, the integrals in question (as in any integral transform) must exist, and be finite. This is the equivalent of saying that the function in question, must be continuous and everywhere differentiable. Second, the function must be “well behaved” at infinity. These two requirements can be seen easier as:

$$(1) \quad \psi(\alpha) = \int_{-\infty}^{\infty} \psi(\beta) d\beta < \infty$$

$$(2) \quad f(\alpha) = 0 \quad \text{as } \alpha \rightarrow \infty$$

Some more complex properties of the Fourier transform, is that we can see that they are linear transforms. This comes into play later in the game, especially in quantum mechanics, which heavily involves the study of linear operators and transforms of the Hilbert space. The last and most complicated property we shall discuss is called The Parsevels relation, which states that if we are given two functions who are integral (Fourier) transforms of one another, then the square of the modulus of those functions will in turn be related by

$$(3) \quad \int_{-\infty}^{\infty} |\psi(\alpha)|^2 d\alpha = \int_{-\infty}^{\infty} |\psi(\beta)|^2 d\beta$$

The proof of which is clear, by substitution of the above definition of the integral transform with the appropriate kernel.

1. DERIVATION OF THE FOURIER TRANSFORM

OK, so we now have the tools to derive formally, the Fourier transform. Once proving one of the Fourier transforms, the change of indexed variables will provide the rest, so without loss of generality, we consider the Fourier transform of time and frequency, given be:

$$(4) \quad f(t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(\omega) e^{-i\omega t} d\omega$$

and we consider some other function, related to the above, by:

$$(5) \quad f(\omega) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

Which is what we hope that we shall arrive at. As previously stated, this comes from the Fourier integrals, where for a given periodic function,

$$f(t) = \frac{a_0}{(2\pi)^{\frac{1}{2}}} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt)$$

where the co-efficients in the above are defined as the even and odd parameters

$$a_n = \frac{1}{L} \int_{-L}^L f(t) \cos\left(\frac{n\pi}{L}t\right) dt, \quad b_n = \frac{1}{L} \int_{-L}^L f(t) \sin\left(\frac{n\pi}{L}t\right) dt$$

Thus, we can sub in the last two equations, the definitions above of a_n and b_n to see that at some time t' ,

$$f(t') = \frac{1}{2L} \int_{-L}^L f(t) dt + \frac{1}{L} \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L}t'\right) \int_{-L}^L f(t) \cos\left(\frac{n\pi}{L}t\right) dt + \frac{1}{L} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}t'\right) \int_{-L}^L f(t) \sin\left(\frac{n\pi}{L}t\right) dt$$

Now, this looks complex, but lets make some resolutions here. We note that the third piece, involves an odd integral, so it will integrate to zero. Now, let us make the substitution, such that

$$\omega = \frac{n\pi}{L}$$

so that we can take the limit as $L \rightarrow \infty$, then equation can be written now as (keeping the odd integral for now)

$$f(t') = \frac{1}{2L} \int_{-L}^L f(t) dt + \frac{1}{\pi} \int_0^{\infty} d\omega \int_{-\infty}^{\infty} f(t) \cos[\omega(t-t')] dt + \frac{1}{\pi} \int_0^{\infty} d\omega \int_{-\infty}^{\infty} f(t) \sin[\omega(t-t')] dt$$

since the sum over ω is a continuous function in the limit as $L \rightarrow \infty$, so we can replace the sum by an integral. Taking this limit also will cause the first term to vanish due to the L in the denominator, and using the condition above that the integral in question actually exists. Now, since the function over ω is even, then we can replace the ω integral as shown in the second and third term by an integral over the entire continuum, only divided by two (see elementary calculus texts for more details). Thus the above is now (without the odd integral):

$$(6) \quad f(t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} f(t) \cos[\omega(t-t')] dt$$

and as we have already stated, the odd integral will yield, after being rewritten with the new limits,

$$(7) \quad 0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} f(t) \sin[\omega(t-t')] dt$$

We now multiply equation (8) by i , and add equations (8) and (9) to see:

$$f(t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} f(t) e^{i\omega(t-t')} dt$$

by making use of equation (1). Thus, we can pull apart the exponentials, and move the t' term, since it is independent of t , to yield:

$$f(t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t'} d\omega \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

now, we can factor out the scaling factor in front, to write the above as:

$$f(t') = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-i\omega t'} \left[\frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt \right] d\omega$$

and we recognize the terms inside the square brackets can be substituted using equation (6) to finally yield:

$$f(t') = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(\omega) e^{-i\omega t'} d\omega$$

as desired (since the t' is just a dummy variable, we can now replace it by t). The proof of the inverse relation is analogous Q.E.D.

Now, as one would expect, since it is the case that we have defined a Fourier transform, as well as the inverse transform, then we should be able to demonstrate that if we start in some basis, then a Fourier transform of a function, followed directly by an inverse transform, should yield the original function. If this were not the case, then these transforms would be put way to the back of our toolbox, and only taken out to destroy a project, rather than enhance it. But, in order to show this calculation, I must give a short explanation of delta functions.

2. DELTA FUNCTIONS AND VERIFICATION OF THE INVERSE TRANSFORM

The delta function, is a mathematical magic trick. In its very basic definition, we see that:

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0, \\ 0, & \text{otherwise,} \end{cases}$$

And they come to use in integrals, since we can exploit the above quality (shifting the origin from 0 to a) as

$$(8) \quad \int_{-\infty}^{\infty} f(x) \delta(x - a) dx = f(a)$$

Now, what makes these seemingly harmless (and at first glance, useless) functions so useful, is best illustrated in an example. In Electrodynamics, one of the first concepts in which we are introduced, is the idea of charge, surface charge and charge density. Now, although in elementary courses, we learn the 1-d, 2-d, and 3-d versions of Coulombs law, in reality, the lower dimensional versions do not make any sense, unless you wanted to sacrifice Gauss' law (which I think most of us would rather give our first born instead). This is because Gauss' law is based ONLY on the charge density, $\rho(\vec{x}) = \rho(x, y, z)$. But if this is the case, how can we define the appropriate charge density in three dimensions for something like a sheet of conducting material with a given charge density $\sigma(x, y)$. The answer is (as you no doubt have guessed) by the use of a delta function. We can see that by the definition above, that we can define a charge density for a sheet of negligible thickness, by the equation:

$$\rho(\vec{x}) = \sigma(x, y) \delta(z)$$

We see that this equation shall follow the requirements, that at any height z off the sheet, that $\rho = 0$ as we would want. So now that we are pros in this subject, we can think about the following family of integrals,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega(x-x')} d\omega$$

Now, by direct integration, we can see that when $x=x'$, then we have the case that the integral becomes unity, but for any other x' , the integral is zero. This sounds awfully familiar. Thus, we see that integrals of the above for can be represented by delta functions as:

$$(9) \quad \delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega(x-x')} d\omega$$

Now that we have this property of delta functions, then we can show that our F.T. and its inverse have the appropriate form. We consider a function $f(t)$ as above, and we shall use the F.T. and its inverse as above, to yield:

$$\begin{aligned} f(t) &= \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(\omega) e^{-i\omega t} d\omega \\ &= \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \left[\frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(t) e^{i\omega t'} dt' \right] d\omega \\ &= \int_{-\infty}^{\infty} dt' \left[\frac{1}{(2\pi)} \int_{-\infty}^{\infty} e^{-i\omega(t-t')} d\omega \right] \\ &= \int_{-\infty}^{\infty} dt' [\delta(t - t')] \\ &= f(t) \end{aligned}$$

Where use has been made of the delta function representation in equation (10) as above.

3. PHYSICAL FOURIER TRANSFORMS

Now that we have seen where the Fourier Transforms come from, listed some properties, and verified that they do indeed transform a from one continuous domain to another, let us list the transforms in which we use in physical systems. The first, and most common, was the transform described above, as the Time/Frequency transform, given by:

$$(10) \quad f(t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(\omega) e^{-i\omega t} d\omega$$

and the inverse transform :

$$(11) \quad f(\omega) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

The next one we encounter, is the relation between the coordinate space vector \vec{x} and a wave propagation vector \vec{k} , for a given wave in 1-D, we can write the transform as:

$$(12) \quad f(\vec{x}) = \int_{-\infty}^{\infty} f(k) e^{i\vec{x} \cdot \vec{k}} dk$$

and the inverse transform :

$$(13) \quad f(k) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

Which we can make a generalization to three dimensions as:

$$(14) \quad \psi(\vec{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} \psi(\vec{k}) e^{i(\vec{k} \cdot \vec{x})} d^3 k$$

and the inverse transform :

$$(15) \quad \psi(\vec{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} \psi(\vec{x}) e^{-i(\vec{k} \cdot \vec{x})} d^3x$$

We note that every time we go up in dimension, we tag on an extra scaling factor of $\frac{1}{2\pi^{\frac{1}{2}}}$. Now, I shall make a note here that we notice that for a given physical quantity, we usually denote its physical presence as $\psi(\vec{x}, t)$, and if we are to transform it, we shall transform it to $\psi(\vec{k}, \omega)$. We see that in the "forward" direction of the transform of $\psi(\vec{x}, t)$ to $\psi(\vec{k}, \omega)$ we see that in the time exponential, there is a negative sign, where in the spatial pieces, there is a lack thereof. This apparent discrepancy shall be discussed at the end of the paper. Now, the usual procedure, for using these transforms in a physical system is as follows:

- (1) Start with a complex problem in the normal coordinates $\psi(\vec{x}, t)$
- (2) Transform the problem to the new coordinates $\psi(\vec{x}, t) \rightarrow \psi(\vec{k}, \omega)$
- (3) Solve the problem in the new coordinates (usually much simpler).
- (4) Transform the answer back to the original coordinates.

This procedure, although sounding long and complex, actually results in a simplification, because it is a definite method of solution, and in most cases, the problem is easier solved in the transformed coordinates. You can almost think of this like relativity. Sometimes, if we switch reference frames, problems in physics become much easier to solve. Thus, with all of this under our belts now, let us compute some common, and useful Fourier Transforms.

4. FOURIER TRANSFORMS IN ELECTRODYNAMICS

We shall stick with the method above, to observe some Fourier transforms. Note that the ones selected for this paper are only due to the wide range of applications of these transforms, but any transform can be computed as directed above. Let us begin with a very simple, but useful function.

1. Let $f(x) = 0$. Thus, we can transform to the K -space using equation (10), to see

$$\begin{aligned} f(k) &= \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \\ &= \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} 0 e^{-ikx} dx \\ &= 0 \end{aligned}$$

So we see (although elementary) that a zero in coordinate space transforms to a zero in k space.

2. Let $f(x) = a\delta(x)$. Here, we shall see what the transform of a delta function looks like in k space:

$$\begin{aligned}
f(k) &= \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(x)e^{-ikx} dx \\
&= \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} a\delta(x)e^{-ikx} dx \\
&= \frac{a}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \delta(x)g(x)dx \\
&= bg(0) \\
&= be^{-i*0} \\
&= b
\end{aligned}$$

Where here we see that the Fourier transform of a delta function is just the constant function 1. Thus, a delta function in coordinate space can be transformed into unity in the k space (a very useful calculation in Electrodynamics).

3. Let $g(x) = \nabla f(x)$ So here, we shall compute the Fourier transform of a derivative function. Note that here we shall need to use the three dimensional representation.

$$\begin{aligned}
f_g(k) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} g(x)e^{-i(\vec{k}\cdot\vec{x})} d^3x \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} \nabla f(x)e^{-i(\vec{k}\cdot\vec{x})} d^3x \\
&= \frac{e^{-i(\vec{k}\cdot\vec{x})}}{(2\pi)^{\frac{3}{2}}} [f(\infty) - f(-\infty)] - \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} \nabla [e^{-i(\vec{k}\cdot\vec{x})}] f(x) d^3x \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} \nabla [e^{-i(\vec{k}\cdot\vec{x})}] f(x) d^3x \\
&= -\frac{ik}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} e^{-i(\vec{k}\cdot\vec{x})} f(x) d^3x \\
&= ikf(k)
\end{aligned}$$

OK, so this calculation is definitely more complex, so let me make some quick notes. We start with the Fourier transform of the desired function. In the first line, we replace $g(x)$ by the derivative function of $f(x)$ as stated earlier. We then make the jump to the next equation using integration by parts, to yield the two terms. We then use the property of the periodic functions (equation 3) to make the first term vanish. We are then left only with the integral term, in which we evaluate the gradient, which shall bring about the factor of $-ik$. We then note that the integral left over is just the Fourier transform of the original function $f(x)$, given by equation . Thus, we see that we can just transform away complicated gradients in the k space, at the cost of only a factor of $-ik$. Pretty sweet deal I think!

4. Let $h(x) = \nabla \cdot g(x) = \nabla^2 f(x)$ Where $g(x)$ is defined as above. Here, we wish to compute the F.T. of the Laplacian. By the line of reasoning above, it is easily shown that:

$$\begin{aligned}
h_g(k) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} h(x) e^{-i(\vec{k}\cdot\vec{x})} d^3x \\
&= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} \nabla^2 f(x) e^{-i(\vec{k}\cdot\vec{x})} d^3x \\
&= i^2 k f(k) \\
&= -k^2 f(k)
\end{aligned}$$

The above Fourier transforms are used in many applications in Electrodynamics, now lets consider a specific example.

Let us consider the three dimensional wave equation. Typically, this equation can be solved by finding an appropriate Green's function, which satisfies:

$$\nabla^2 G(\vec{x}, t) - \frac{1}{c^2} \frac{\partial^2 G(\vec{x}, t)}{\partial t^2} = -4\pi\delta(\vec{x})\delta(t)$$

Now, we can solve for this greens function using the standard Fourier transform, by transforming away the ∇^2 and then transforming away the the time component. We get, after some partial integration (exact calculation is above) that the transforms correspond to first:

$$-\vec{k}^2 G(\vec{k}, t) - \frac{1}{c^2} \frac{\partial^2 G(\vec{k}, t)}{\partial t^2} = -4\pi\delta(t)$$

from the spatial transform using (16), and then using (12) to transform away the time, yields:

$$-\vec{k}^2 G(\vec{k}, \omega) + \frac{\omega^2}{c^2} G(\vec{k}, \omega) = -4\pi$$

from which we can now solve for the transformed Green's function as:

$$G(\vec{k}, \omega) = \frac{4\pi}{k^2 - \frac{\omega^2}{c^2}}$$

Which is now the Green's function in our k space, which is now a simple, algebraic equation. We can thus take this now simple equation and transform it back using the two inverse transforms (15 and 11), to finish the job.

5. FOURIER TRANSFORMS IN QUANTUM MECHANICS

Fourier Transforms play an immensely important role in quantum mechanics. In quantum mechanics, we study mostly wave functions of particles. Since the kernel of Fourier transforms deals with waves, they provide us with a way to transform from one basis of functions to another. Now this is key, since in quantum mechanics, we deal with different eigenfunctions of an eigenvalue equation, where the eigenstates are the wave functions. Since we can represent the eigenvalue equation of quantum mechanics (i.e. The Schrodinger Equation) in both a complete set of coordinate eigenstates, by solving the familiar Schrodinger Equation:

$$\frac{-\hbar^2}{2m} \nabla^2 \Psi(\vec{x}, t) + V(\vec{x})\Psi(\vec{x}, t) = i\hbar \frac{\partial \Psi(\vec{x}, t)}{\partial t}$$

or we can take an alternate route, by solving for the eigenstates of the momentum, representation of the Schrodinger equation:

$$\frac{p^2}{2m}\Psi(\vec{x}, t) + V(\vec{x})\Psi(\vec{x}, t) = i\hbar\frac{\partial\Psi(\vec{x}, t)}{\partial t}$$

By making the canonical substitution,

$$\vec{p} = -i\hbar\nabla$$

We shall see in a moment that solving either representation gives us all the information about both representations, by the use of Fourier Transforms. But let us ponder for a moment about the above relation of the momentum. In classical physics, we are taught that

$$\vec{p} = \vec{p}(t) = m\frac{dx}{dt}$$

Where the momentum is clearly a function of time. In the above canonical transform, we see that the new definition of p is completely independent of t. How can this be, and why are we justified in making this substitution? We now have enough information to show where this innocuous relation comes from.

Let us recall that De'Broglie, defined the matter waves of all matter particles, following suit of Einstein defining the wave particle duality of nature. De'Broglie defined these waves by the momentum relation:

$$p = \frac{h}{\lambda}$$

Now recalling that for any wave, we can define the wave number as $k = \frac{2\pi}{\lambda}$. Thus, we can multiply the above by $\frac{2\pi}{2\pi}$ to see that we can rewrite the De'Broglie relation as (with $\hbar = \frac{h}{2\pi}$):

$$p = \hbar k$$

From this, we can now solve for $k = \frac{p}{\hbar}$. Thus, we can use our above original Fourier Transforms (equations), to solve for the Quantum Mechanical Fourier Transform, which shall take us from the coordinate space basis, into the momentum space basis, which will give us the relation between the eigenstates (wavefunctions) and thus the solutions of the two isomorphic representations. So, we can then make the denoted change of variables, with $k = \frac{p}{\hbar}$ and hence $dk = \frac{dp}{\hbar}$ and sub these into equations to yield:

$$(16) \quad f(x) = \frac{1}{(2\pi\hbar)^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(p)e^{i\frac{p}{\hbar}x} dp$$

and the inverse transform :

$$(17) \quad f(p) = \frac{1}{(2\pi\hbar)^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(x)e^{-i\frac{p}{\hbar}x} dx$$

The Desired quantum mechanical Fourier Transforms. We can now make a calculation similar to one of the above, by taking the gradient of both sides of equation

to see:

$$\begin{aligned}
\nabla(f(x)) &= \nabla\left(\frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int_{-\infty}^{\infty} f(p)e^{i\frac{p}{\hbar}x} dp\right) \\
&= \frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int_{-\infty}^{\infty} \nabla(f(p))e^{ikx} dp + \frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int_{-\infty}^{\infty} f(p)\nabla(e^{i\frac{p}{\hbar}x})dp \\
&= \frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int_{-\infty}^{\infty} \nabla(e^{i\frac{p}{\hbar}x})f(p)dp \\
&= \frac{ip}{\hbar}\frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int_{-\infty}^{\infty} f(p)e^{i\frac{p}{\hbar}x} dp \\
&= \frac{ip}{\hbar}f(x)
\end{aligned}$$

Thus, we can cancel out the test function $f(x)$ on both sides of the equation, to yield the desired relation,

$$\vec{p} = -i\hbar\nabla$$

And we see that we have not been lied to, that this bizarre relation actually holds. Thus, the Fourier transform provided us with a means of finding out the momentum relation to coordinate space.

6. AN EXTENDED APPLICATION

Now, as promised, I would like to briefly discuss the switch in the sign of the exponential in the "forward directions" of the Fourier transforms (see equations 16 and 12, with inverse pairs 15 and 11). We would think that in one direction, the sign should stay constant. But this is actually not the case. This can be better explored by defining a four dimensional Fourier transform, to account for both the time and space components being transformed at once. When carrying out this procedure, one can compose the exponentials to yield the single transform:

$$\psi(\vec{x}, t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} e^{i[(\vec{k}\cdot\vec{x}) - \omega t]} \psi(\vec{k}, t) d^3k d\omega$$

Which we can see we have spatial coordinates and time coordinates, not mixing together. This sounds strangely enough like a dot product. Thus, we can see that a four dimensional Dot product, just happens to carry an innate minus sign with it, due to the Minkowski space time metric:

$$\mathbf{g}_{\nu\mu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, using this metric and the fact that the four dimensional dot product can be computed as (for two four vectors x^μ and k_ν):

$$\begin{aligned}
x^\mu k_\nu &= \mathbf{g}_{\mu\nu} x^\mu k^\nu \\
&= -(x^0 k^0) + x^1 k^1 + x^2 k^2 + x^3 k^3 \\
&= -ctk_0 + xk_x + yk_y + zk_z \\
&= -ctk_0 + (\vec{k} \cdot \vec{x}) \\
&= -\omega t + (\vec{k} \cdot \vec{x}) \\
&= (\vec{k} \cdot \vec{x}) - \omega t
\end{aligned}$$

which is the exact term in our composed transform. Thus, we can define in four vector notation, the four vector Fourier transform:

$$(18) \quad \psi(x^\alpha) = \psi(\vec{x}, t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} e^{i\mathbf{g}_{\mu\nu} x^\mu k^\nu} \psi(k^\alpha) d^4 k$$

and the inverse transform as:

$$(19) \quad \psi(k^\alpha) = \psi(\vec{k}, \omega) = \int_{-\infty}^{\infty} e^{-i\mathbf{g}_{\mu\nu} x^\mu k^\nu} \psi(x^\alpha) d^4 x$$

Which are the Fourier Transforms we can use for both Special and General Relativity. For more information on the above, please see O'Brien and Cox, Four Dimensional Fourier Transforms, 2004.