Lecture 23-24

4/18/2023
4/20/2023

Motion of Relativistic Particles in the EM field

recap:

Classical motion → classical Lagrangian

$$S_{cl}^{ac} = \int L_{cl}\, dt$$
$$L_{cl} = \frac{mv^2}{2} + \frac{e}{c}\vec{A}\cdot\vec{v} - e\varphi$$

Relativistic action $\boxed{S}$ for a free motion:

$$dS^{ac} = c\,d\tau = c\,dt\sqrt{1-\frac{v^2}{c^2}}$$

$$S_f^{ac} = -\int_1^2 \alpha \cdot ds = -\int mc\, ds$$

$ds^2 = dx_i dx^i$ relativistic invariant

$\alpha = mc$ ← from the classical limit

$$S_{cl}^{ac} = S_f^{ac} + \int \frac{e}{c}\vec{A}\vec{v}dt - \int e\varphi dt$$
$$dS_{in}^{ac} = \frac{e}{c}\vec{A}\,d\vec{r} - \frac{e}{c}\varphi \overbrace{dt\cdot c}^{dx^0}$$

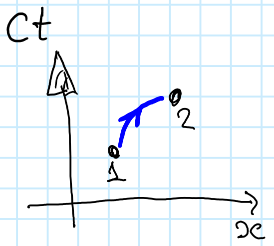
$$dS_{in}^{ac} = -\frac{e}{c}\left(\varphi\cdot dx^0 - \vec{A}d\vec{r}\right) =$$
$$= -\frac{e}{c}A_i dx^i$$

$$\boxed{dS_{in}^{ac} = -\frac{e}{c}A_i dx^i \qquad A^i = (\varphi, \vec{A})}$$

4D-vector potential
$$A^i = (\varphi, \vec{A})$$

We have to include the relativistic invariant describing interaction between charge particle and E-M field:

$$S^{ac} = S_f^{ac} + S_{int}^{ac} = \int_1^2 \left(-mc\,ds - \frac{e}{c}A_i dx^i\right)$$



$$S^{ac} = \int_1^2 L\cdot dt = \int_1^2\left(-mc^2\sqrt{1-\frac{v^2}{c^2}} + \frac{e}{c}\vec{A}\frac{d\vec{r}}{dt} - e\varphi\right)dt$$

Relativistic formula for the Lagrangian:

$$\boxed{L = -mc^2\sqrt{1-v^2/c^2} + \frac{e}{c}\vec{A}\vec{v} - e\varphi}$$

Lagrangian Mechanics:

$$P = \frac{\partial L}{\partial \dot{q}}\ ; \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

Generalized momentum

Relativistic momentum:

$$\boxed{\vec{P} = \vec{\nabla}_v L = \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{e}{c}\vec{A}}$$

Hamiltonian → Relativistic particle in EM-field:

$$H = \vec{v}\frac{\partial L}{\partial \vec{v}} - L = \frac{mv^3}{\sqrt{1-\frac{v^2}{c^3}}} + \cancel{\frac{e}{c}\vec{A}\vec{v}} + mc^2\sqrt{1-\frac{v^2}{c^2}} - \cancel{\frac{e}{c}\vec{A}\vec{v}} + e\varphi$$

$$H = \dot{q}\frac{\partial L}{\partial \dot{q}} - L$$

$$H = \text{const.} \equiv \text{energy}$$

$$\boxed{H = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} + e\varphi}$$

Classical results:

$$\bar{P} = \vec{p} + \frac{e}{c}\vec{A}$$
$$\vec{p} \equiv m\vec{v}$$
$$\mathcal{E} = \frac{mv^2}{2} = \frac{(\bar{P} - \frac{e}{c}\bar{A})^2}{2m}$$

Lagrangian: $$\boxed{L = -mc^2\sqrt{1-\frac{v^2}{c^2}} - e\varphi + \frac{e}{c}\vec A\vec v}$$

$$\boxed{H = \vec v\frac{\partial L}{\partial \vec v} - L = \frac{mc^2}{\sqrt{1-v^2/c^2}} + e\varphi}$$

$$\boxed{\vec P = \vec p + \frac{e}{c}\vec A}$$

$\vec P$ - canonical momentum

$\vec p = \frac{m\vec v}{\sqrt{1-v^2/c^2}}$ - kinematic momentum

The Hamiltonian can be expressed as a function of $\vec P$

$$\boxed{H = \sqrt{m^2c^2 + c^2\left(\vec P - \frac{e}{c}\vec A\right)^2} + e\varphi}$$

Relativistic equation of motion of charged particles in EM field

The Lagrange Equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec v}\right) \equiv \frac{d\vec P}{dt} = \frac{\partial L}{\partial \vec r}$$

$$\boxed{L = -mc^2\sqrt{1-\frac{v^2}{c^2}} - e\varphi + \frac{e}{c}\vec A\vec v}$$

$$\vec P = \vec p + \frac{e}{c}\vec A \qquad \boxed{\frac{\partial L}{\partial \vec v} = \vec P}$$

kinematic momentum $\boxed{\vec p = \frac{m\vec v}{\sqrt{1-\frac{v^2}{c^2}}}}$

$$\vec\nabla\cdot L = \frac{\partial L}{\partial \vec r} = -e\vec\nabla\varphi + \frac{e}{c}\vec\nabla(\vec A\cdot\vec v) =$$

$$= e\vec E_{st} + \frac{e}{c}(\vec v\cdot\vec\nabla)\vec A + \frac{e}{c}\overset{0}{\cancel{(\vec A\cdot\vec\nabla_v)\vec v}} + \frac{e}{c}\vec v\times\overbrace{(\vec\nabla\times\vec A)}^{\vec B} + \frac{e}{c}\vec A\times\overset{0}{\cancel{(\vec\nabla\times\vec v)}}$$

$\vec E_{st} = -\vec\nabla\varphi$ electrostatic field

$$\vec\nabla(\underset{\vec A\cdot\vec v}{\vec a\cdot\vec b}) = (\underset{\vec v}{\vec b}\cdot\vec\nabla)\underset{\vec A}{\vec a} + \underset{\text{"0"}}{(\underset{\vec A}{\vec a}\cdot\vec\nabla)\underset{\vec v}{\vec b}} + \underset{\vec v}{\vec b}\times(\vec\nabla\times\underset{\vec A}{\vec a}) + \underset{\text{"0"}}{\underset{\vec A}{\vec a}\times(\vec\nabla\times\underset{\vec v}{\vec b})}$$ ← (Vector analysis)

$$\frac{d}{dt}\vec P = -e\vec\nabla\varphi + \frac{e}{c}(\vec v\times\vec B) + \frac{e}{c}(\vec v\cdot\vec\nabla)\vec A$$

This equation is not a relativistic invariant

kinematic $\vec p$

$$\frac{d}{dt}\left(\vec p + \frac{e}{c}\vec A\right) = -e\vec\nabla\varphi + \frac{e}{c}(\vec v\times\vec B) + \frac{e}{c}(\vec v\ \vec\nabla)\vec A$$

$$\frac{d\vec A}{dt} = \frac{\partial\vec A}{\partial t} + (\vec v\cdot\vec\nabla)\vec A$$

For the particle $\vec r(t) \Rightarrow \vec A(\vec r(t), t)$

$$\frac{d}{dt}\vec p = -e\vec\nabla\varphi - \frac{e}{c}\frac{\partial\vec A}{\partial t} + \frac{e}{c}(\vec v\times\vec B) + \frac{e}{c}(\vec v\cdot\vec\nabla)\vec A - \frac{e}{c}(\vec v\cdot\vec\nabla)\vec A$$

kinematic momentum $\vec p = \frac{m\vec v}{\sqrt{1-v^2/c^2}}$

$$\boxed{\frac{d}{dt}\vec p = e\vec E + \frac{e}{c}(\vec v\times\vec B)}$$

Lorentz Force

$$\vec E = -\frac{1}{c}\frac{\partial\vec A}{\partial t} - \vec\nabla\varphi$$

Changes of the kinetic energy $\mathcal{E}$:

$$\frac{d\mathcal E}{dt} = \frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-v^2/c^2}}\right) = \frac{d}{dt}\left(\sqrt{m^2c^4 + p^2c^2}\right) = \frac{2\vec p c^2\frac{d\vec p}{dt}}{2\sqrt{m^2c^4 + p^2c^2}} =$$

$$= \frac{c^2\ \vec p\ \frac{d\vec p}{dt}}{\mathcal E} = \vec v\frac{d\vec p}{dt} \Longrightarrow \boxed{\frac{d\mathcal E}{dt} = \vec v\frac{d\vec p}{dt}}$$

$$\frac{d\mathcal E}{dt} = \vec v\cdot\left(e\vec E + \frac{e}{c}(\vec v\times\vec B)\right) = \vec v e\vec E + \frac{e}{c}\underset{0}{\cancel{\vec v(\vec v\times\vec B)}} \dashrightarrow \boxed{\frac{d\mathcal E}{dt} = \vec v e\vec E}$$