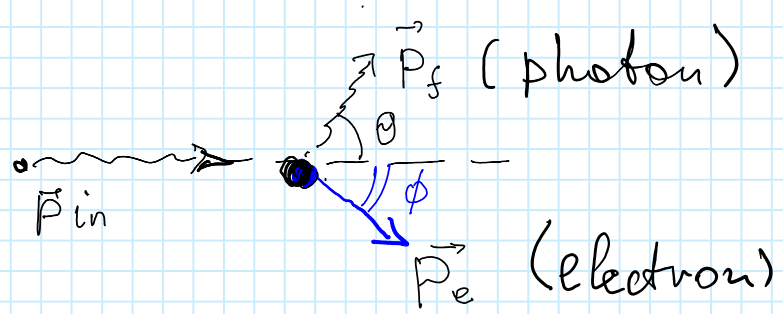
Lecture 22

04/13/23

The Compton Effect

collisions between electrons and photons

scattering of EM-waves



Photon: $\varepsilon_{ph} = p_i \cdot c$; ($m_{photon} = 0$)

Electron: $\varepsilon_e = m_e c^2$

$v_e = 0$

Calculate the energy of a scattered photon as a function of the scattering angle θ.

4-vector of momentum

(1) before collision

$$P^i_{initial} = P^i_{in,\,phot} + P^i_{in,e} = \Big(\underbrace{\frac{\varepsilon^{in}_{ph}}{c}, \vec{p}^{\,in}_{ph}}_{photon}\Big) + \Big(\underbrace{\frac{m_e \cdot c^2}{c}, 0}_{electron}\Big)$$

$\boxed{initial \equiv in}$

$$P^i_{in} = \left(\frac{\varepsilon^{in}_{ph} + m_e c^2}{c}, \vec{p}^{\,in}_{ph}\right)$$

(2) after collision (scattering)

$\boxed{\text{outgoing photon} \equiv out}$

$$P^i_{out} = \left(\frac{\varepsilon_e + \varepsilon^{out}_{ph}}{c}, \vec{p}^{\,out}_{ph} + \vec{p}_e\right)$$

$$P^i_{in} = P^i_{out} \Rightarrow \begin{cases} \varepsilon^{in}_{ph} + m_e c^2 = \varepsilon^{out}_{ph} + \varepsilon_e \\ \vec{p}^{\,in}_{ph} = \vec{p}^{\,out}_{ph} + \vec{p}_e \end{cases}$$

$$\begin{cases} \varepsilon_e = \varepsilon^{in}_{ph} - \varepsilon^{out}_{ph} + m_e c^2 \\ \vec{p}_e = \vec{p}^{\,in}_{ph} - \vec{p}^{\,out}_{ph} \end{cases}$$

$$\underline{\left(\frac{\varepsilon_e}{c}\right)^2 - p_e^2 = m_e^2 c^2} = \left(\frac{\varepsilon^{in}_{ph} - \varepsilon^{out}_{ph} + m_e c^2}{c}\right)^2 - (\vec{p}^{\,in}_{ph} - \vec{p}^{\,out}_{ph})^2$$

$$\underbrace{\left(\frac{\varepsilon^{i}_{ph}}{c}\right)^2 - p^{in\,2}_{ph}}_{m_{ph} \cdot c^2 = 0} + \underbrace{\left(\frac{\varepsilon^{out}_{ph}}{c}\right)^2 - p^{out\,2}_{ph}}_{0} + \cancel{m_e^2 c^2} + 2 m_e c^2 \frac{\varepsilon^{in}_{ph} - \varepsilon^{out}_{ph}}{c^2} - 2\frac{\varepsilon^{in}_{ph}\varepsilon^{out}_{ph}}{c^2} +$$

$$+ 2\vec{p}^{\,in}_{ph} \cdot \vec{p}^{\,out}_{ph} = \cancel{m_e^2 c^2}$$

Eq. (*)

$$m_e c^2 \frac{(\varepsilon_{ph}^{in} - \varepsilon_{ph}^{out})}{c^2} + p_{ph}^{in} \cdot p_{ph}^{out} \cdot \cos\theta - \frac{\varepsilon_{ph}^{in} \cdot \varepsilon_{ph}^{out}}{c^2} = 0$$

$$\boxed{p_{ph} = \frac{\varepsilon_{ph}}{c}}$$

$$\Rightarrow m_e c^2 \cdot \varepsilon_{ph}^{in} - m_e c^2 \varepsilon_{ph}^{out} - \varepsilon_{ph}^{in} \cdot \varepsilon_{ph}^{out} (1 - \cos\theta) = 0$$

$$\Rightarrow \varepsilon_{ph}^{out} \left(m_e c^2 + \varepsilon_{ph}^{in} (1 - \cos\theta) \right) = m_e c^2 \varepsilon_{ph}^{in}$$

$$\Rightarrow \boxed{\varepsilon_{ph}^{out} = \varepsilon_{ph}^{in} \frac{m_e c^2}{m_e c^2 + \varepsilon_{ph}^{in} (1 - \cos\theta)}}$$

The Compton formula can be written in more simple form, if the photon wavelength λ is introduced. λ is the wavelength of the EM radiation

$$\lambda = T \cdot c = \frac{2\pi}{\omega} c = \frac{2\pi}{k};$$

photon's energy and momentum $\boxed{\varepsilon = \hbar \omega \quad p = \frac{\varepsilon}{c} = \hbar \frac{\omega}{c} = \hbar k = \frac{h}{\lambda}}$

$\hbar = \frac{h}{2\pi}$ ← reduced Planck's constant

h – the Planck constant

From Eq. (*): using $p_{ph} = \frac{\varepsilon_{ph}}{c}$

$$m_e c \left(\frac{1}{p_{ph}^{out}} - \frac{1}{p_{ph}^{in}} \right) - (1 - \cos\theta) = 0$$

$$\frac{1}{p_{ph}} = \frac{1}{\hbar k} = \frac{\lambda}{h} \Rightarrow \frac{m_e c}{h} (\lambda_{out} - \lambda_{in}) = 1 - \cos\theta$$

$$\lambda_{out} - \lambda_{in} = \frac{h}{m_e c} (1 - \cos\theta)$$

We can introduce a new physical constant

$\boxed{\lambda_e = \frac{h}{m_e c}}$ ← the Compton wavelength of electron

$$\boxed{\lambda_{out} = \lambda_{in} + \lambda_e (1 - \cos\theta)}$$

$$\lambda_e = 2.42 \times 10^{-12} m = 2.42 * 10^{-2} \text{Å}$$

Positronium annihilation.

Positronium energy:

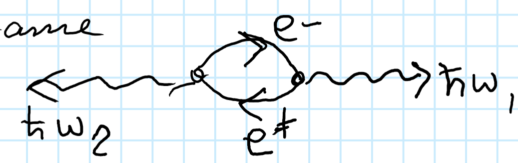
$m_{e^-} = m_{e^+} = m$

$$E = 2mc^2 - \frac{\mu e^4}{2\hbar^2} \cong 2mc^2 - \frac{me^4}{4\hbar^2}$$

$\mu = \frac{m}{2}$ the reduced mass

small fraction of mc^2

CM-frame



$\hbar\omega_1 = \hbar\omega_2$

$$2\hbar\omega \cong 2mc^2 - \frac{me^4}{4\hbar^2} \Rightarrow \boxed{\hbar\omega = mc^2 - \frac{me^4}{8\hbar^2}}$$

In the CM frame:

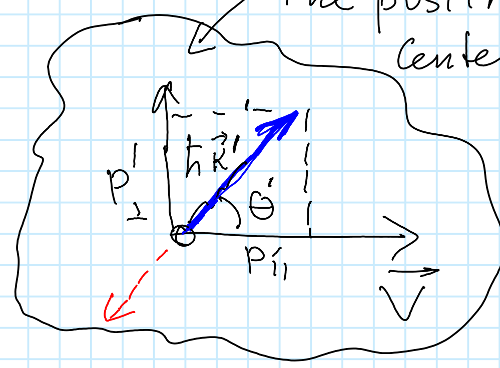
2 equivalent photons, because the total momentum $\vec{P}_{tot} = 0$

Photon energy $\varepsilon = \hbar\omega$:

$$\boxed{P^i_{ph} = \left(\frac{\varepsilon}{c}, \hbar\vec{k}\right)}$$

$$\boxed{P^i_{ph} \cdot P_{ph\,i} = 0}$$

The positronium Center of Mass (CM) frame



$k = \omega/c$

$k' = \omega'/c$

Lab. Frame:

Lorentz Transformation

$$\frac{\varepsilon}{c} = \frac{\varepsilon'/c + \frac{V}{c}p'_{\parallel}}{\sqrt{1-\frac{V^2}{c^2}}} = \frac{\hbar k'\left(1 + \frac{V}{c}\cos\theta'\right)}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$\Rightarrow \boxed{\varepsilon = \hbar\omega' \frac{1 + \frac{V}{c}\cos\theta'}{\sqrt{1-\frac{V^2}{c^2}}}}$$

$$p_{\parallel} = \hbar k_{\parallel} = \hbar k\cos\theta = \frac{\hbar k'_{\parallel} + \frac{V}{c}\frac{\varepsilon'}{c}}{\sqrt{1-V^2/c^2}} = \frac{\hbar\omega'}{c}\,\frac{\cos\theta' + \frac{V}{c}}{\sqrt{1-V^2/c^2}}$$

The Lab. Frame angle θ

$p = \frac{\varepsilon}{c} = \hbar k$

$$\cos\theta = \frac{p_{\parallel}}{p} = \frac{V + \cos\theta'}{1 + V\cos\theta'}, \Rightarrow \boxed{\cos\theta = \frac{\cos\theta' + V}{1 + V\cos\theta'}}, \quad c = 1$$

for $c=1$

$$\boxed{\cos\theta' = \frac{\cos\theta - V}{1 - V\cos\theta}} \quad c = 1$$

$$\boxed{W(\theta) = \frac{\left(1 - V^2/c^2\right)}{4\pi\left(1 - \frac{V\cos\theta}{c}\right)^2}}$$

$$\frac{d\cos\theta'}{d\cos\theta} = \frac{1}{1 - V\cos\theta} + \frac{(\cos\theta - V)V}{(1-V\cos\theta)^2} = \frac{\cancel{V\cos\theta} - V^2 + 1 - \cancel{V\cos\theta}}{(1-V\cos\theta)^2} = \frac{1 - V^2}{(1 - V\cos\theta)^2}$$

$$W'(\theta')\,d\Omega' = W(\theta)\,d\Omega$$

$$W(\theta) = W'(\theta')\cdot\frac{\sin\theta' d\theta'}{\sin\theta\, d\theta} = \frac{1}{4\pi}\,\frac{d(\cos\theta')}{d(\cos\theta)}$$

$W'(\theta') = \frac{1}{4\pi}$

$W(\theta')$ and $W(\theta)$ are photon angular distribution functions in the $e^- + e^+$ -proper and laboratory frames.