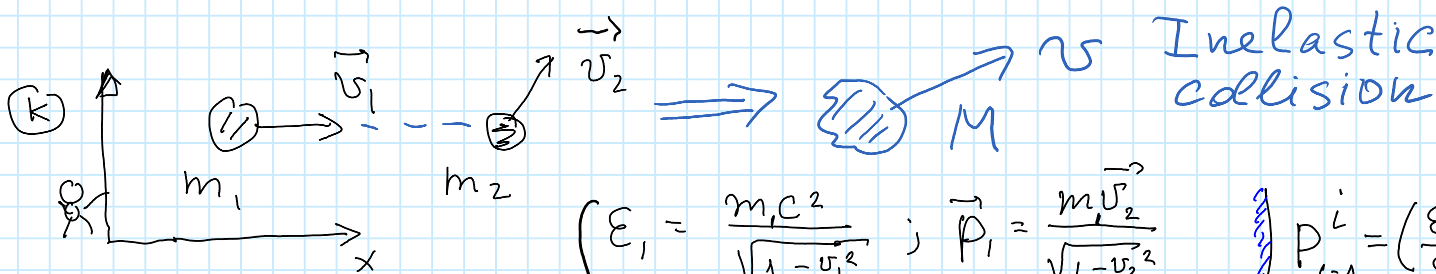
Lecture 21

4/11/23

Example 2:

system of two particles



In (K) frame

$$\begin{cases} \varepsilon_1 = \dfrac{m_1 c^2}{\sqrt{1-\frac{v_1^2}{c^2}}}; & \vec{p}_1 = \dfrac{m_1\vec{v}_2}{\sqrt{1-\frac{v_2^2}{c^2}}} \\ \varepsilon_2 = \dfrac{m_2 c^2}{\sqrt{1-\frac{v_2^2}{c^2}}}; & p_2 = \dfrac{m_2 \vec{v}_2}{\sqrt{1-\frac{v_2^2}{c^2}}} \end{cases}$$

$$p^i_{\alpha=1} = \left(\frac{\varepsilon_1}{c}, \vec{p}_1\right)$$
$$p^i_{\alpha=2} = \left(\frac{\varepsilon_2}{c}, \vec{p}_2\right)$$

$$\boxed{p^i_{total} = \left(\frac{\varepsilon_1+\varepsilon_2}{c}, \vec{p}_1+\vec{p}_2\right)}$$

$$\boxed{p^i_{total}\cdot p_{i\,total} = \left(\frac{\varepsilon_1+\varepsilon_2}{c}\right)^2 - (\vec{p}_1+\vec{p}_2)^2 = = \text{invariant} \equiv M\cdot c^2}$$

The rest mass of a new particle M:

$$M^2c^2 = \left(\frac{\varepsilon_1+\varepsilon_2}{c}\right)^2 - (\vec{p}_1+\vec{p}_2)^2 = \underbrace{\frac{\varepsilon_1^2}{c^2} - p_1^2}_{m_1^2c^2} + \underbrace{\frac{\varepsilon_2^2}{c^2} - p_2^2}_{m_2^2c^2} + 2\left(\frac{\varepsilon_1\varepsilon_2}{c^2} - \vec{p}_1\cdot\vec{p}_2\right)$$

The mass M of a new particle:

$$M^2 = \frac{1}{c^2}\left[m_1^2c^2 + m_2^2c^2 + 2\frac{m_1m_2c^2\left(1-\frac{\vec{v}_1\cdot\vec{v}_2}{c^2}\right)}{\sqrt{\left(1-\frac{v_1^2}{c^2}\right)}\cdot\sqrt{\left(1-\frac{v_2^2}{c^2}\right)}}\right]$$

$$M = \left[m_1^2 + m_2^2 + 2m_1m_2 + 2m_1m_2\left(\frac{1-\frac{\vec{v}_1\vec{v}_2}{c^2}}{\sqrt{1-\frac{v_1^2}{c^2}}\cdot\sqrt{1-\frac{v_2^2}{c^2}}} - 1\right)\right]^{1/2}$$

$$\boxed{M = \left[(m_1+m_2)^2 + 2m_1m_2\left(\frac{1-\frac{\vec{v}_1\cdot\vec{v}_2}{c^2}}{\sqrt{\left(1-\frac{v_1^2}{c^2}\right)\left(1-\frac{v_2^2}{c^2}\right)}} - 1\right)\right]^{1/2}}$$

Simple case: $\vec{v}_2 = 0$;

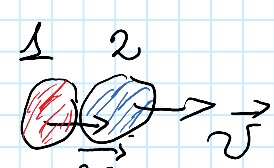
$M \neq m_1 + m_2$

$$\boxed{M = \left[(m_1+m_2)^2 + 2m_1m_2\left(\frac{1}{\sqrt{1-\frac{v_1^2}{c^2}}} - 1\right)\right]^{1/2}}$$

$M > m_1 + m_2$

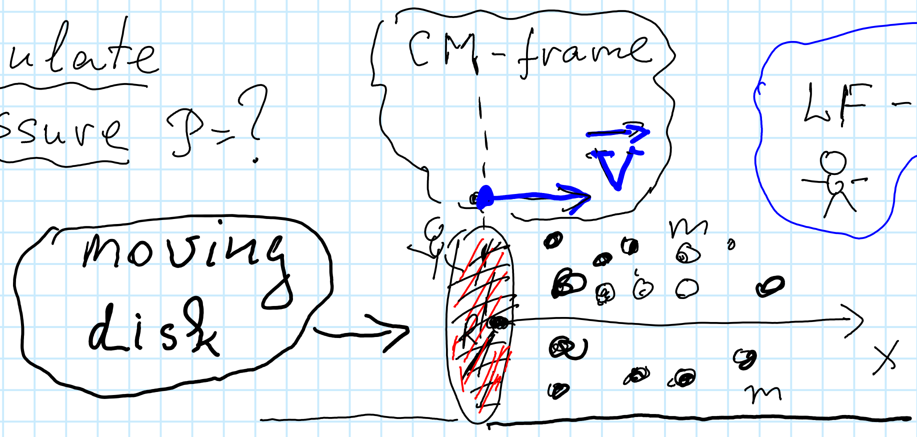
Simplest case: $\vec{v}_1 = \vec{v}_2 = \vec{v}$

$$M = \left[(m_1+m_2)^2 + 2m_1\cdot m_2\underbrace{\left(\cancel{\frac{1-\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}} - 1\right)}_{0}\right] = m_1 + m_2$$



Problem "Pressure"

Calculate pressure $\mathcal{P} = ?$



LF - Laboratory Frame

$$\boxed{\mathcal{P} = \frac{F}{S}}$$

$$\boxed{\vec{F} = \frac{d\vec{p}}{dt}}$$

Given Values:

Radius: R_0

Velocity: v

Particles: m_0, n_0

density $\mathcal{S}$

Determination of $d\vec{p}$

Momentum transfer $d\vec{p}_1$ induced by collision of particles in the Laboratory Frame

$$\boxed{\mathcal{P} = \frac{|\Delta\vec{p}| \cdot (\text{Frequency of collisions}) \cdot \cancel{\Delta t}}{\cancel{\Delta t} \cdot S}}$$

In the disk frame (CM-frame):

the initial velocity of particles $v_i' = -V$ and final velocity is $v_f' = V$.

After Collision

The particle velocity v_f in the Lab. Frame?

$$\left(v = \frac{v' + V}{1 + \frac{v' V}{c^2}} \right) \Rightarrow \boxed{v_f = \frac{v_f' + V}{1 + v_f' V / c^2} = \frac{2V}{1 + \frac{V^2}{c^2}}}$$

Initial momentum $\vec{p}_i = 0$ (Lab. Frame)

Final momentum $\vec{p}_f$:

$$\boxed{p_f = \frac{m_0 v_f}{\sqrt{1 - \frac{v_f^2}{c^2}}}}$$

$$\boxed{\Delta p = p_f - \cancel{p_i}^{\,0} = \frac{m_0 \, 2V/(1 + \frac{V^2}{c^2})}{\left[1 - \frac{4V^2}{(1+\frac{V^2}{c^2})^2 c^2}\right]^{1/2}} = \frac{2 m_0 V}{(1 - \frac{V^2}{c^2})}}$$

The force:

$$\mathcal{F} = \frac{\Delta p}{\Delta t} \underbrace{(\text{number of collisions})}_{V \cdot \Delta t \cdot S \cdot n_0} = \frac{\Delta p \, V \cdot \cancel{\Delta t} \cdot S \cdot n_0}{\cancel{\Delta t}} = \Delta p V S n_0$$

Pressure:

$$\boxed{\mathcal{P} = \frac{\mathcal{F}}{S} = \Delta p V \cdot n_0 = 2 \frac{m_0 V^2}{1 - \frac{V^2}{c^2}} \cdot n_0}$$

Pressure is a relativistic invariant. (Calculations in the disk frame.)

The particle velocity is $v_p' = -V$.

the disk frame:

$$p_{in}' = \frac{m_0 v_p'}{\sqrt{1 - v_p'^2/c^2}} = \frac{m_0 V}{\sqrt{1 - V^2/c^2}}$$

Particle's momentum transfer

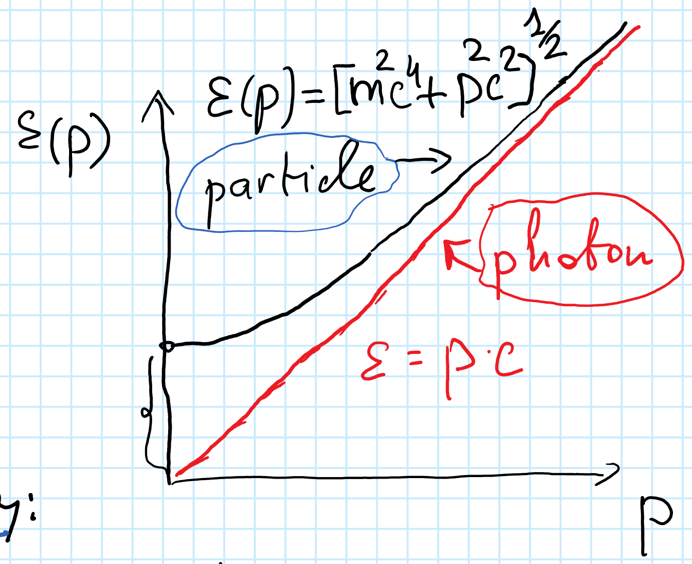
$$\Delta p' = 2 p_{in} = \frac{2 m_0 V}{\sqrt{1 - V^2/c^2}}$$

Particle's density in the disk frame: $n' = n_0 / \sqrt{1 - V^2/c^2}$

Pressure:

$$\boxed{\mathcal{P}' = \frac{\Delta p'}{\cancel{\Delta t'}} \frac{\overset{\vec{v} = -\vec{V}}{v} \cdot \cancel{\Delta t'} \cdot \pi R^2 \cdot n_0}{\cancel{\pi R^2} \sqrt{1 - \frac{V^2}{c^2}}} = \frac{2 m_0 V^2 \cdot n_0}{(1 - \frac{V^2}{c^2})}} \Rightarrow \boxed{\mathcal{P} = \mathcal{P}'}$$

Photons



Particle with $\boxed{m=0}$

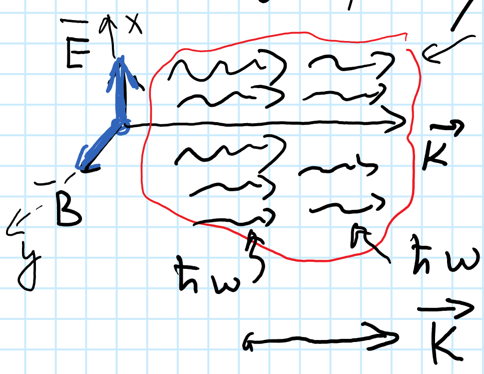
The energy: $\varepsilon(p)=\sqrt{m^2c^4+p^2c^2}=pc$

$\varepsilon_{ph}(p)=p\cdot c$; Photon energy: $\boxed{\varepsilon(p)=\hbar\omega}$ ω – the frequency of EM wave

"$\hbar$" – the reduced Planck constant

EM waves are fluxes of the massless particles, photons.

Photon momentum: photon momentum depends on the frequency "ω" of the EM wave: $\boxed{\vec{p}=\hbar\frac{\omega}{c}\cdot\hat{e}_k=\hbar k\,\hat{e}_k}$



This is the flux of photons.

wave number $\boxed{k=\frac{\omega}{c}=\frac{2\pi}{\lambda}}$

Poynting vector: $\vec{S}=c\,\underbrace{(w_E+w_B)}\,\hat{e}_k=c\,w\,\hat{e}_k$

$\vec{S}=\frac{\vec{E}\times\vec{B}}{\mu_0}$ classical expression

The flux of photons

The energy density of the EM field w: $\boxed{w=\hbar\omega\cdot n}$

n is the volume density of photons.

$$\vec{S}=\hbar\omega\cdot n\cdot c\cdot\hat{e}_k=\varepsilon_{ph}(p)\cdot n\cdot c\,\hat{e}_k=\hbar\overset{(\omega)}{k}\cdot c\,n\,c\cdot\hat{e}_k\equiv p\cdot c^2\cdot n\,\hat{e}_k$$

Photon's 4D-vector: $\boxed{P^i_{ph}=\left(\frac{\varepsilon}{c},\vec{p}\right)\equiv\left(\frac{\hbar\omega}{c},\hbar\vec{k}\right)\equiv\left(\hbar k,\hbar\vec{k}\right)}$

$P^i_{ph}\cdot P_{ph_i}=\left(\frac{\varepsilon}{c}\right)^2-p^2=\hbar^2k^2-(\hbar\vec{k})^2=0$ The wave-vector → $\vec{k}=k\cdot\hat{e}_k$

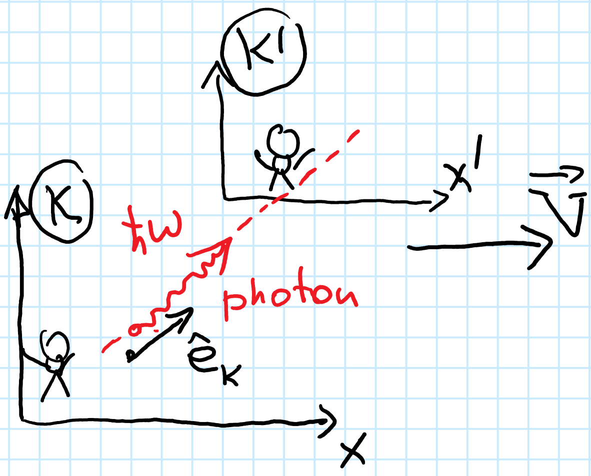
$\varepsilon=\hbar\omega$ and $p=\hbar\frac{\omega}{c}=\hbar k$

Lorentz tranformation for photons:

$$\begin{cases}\varepsilon=\dfrac{\varepsilon'+V\cdot p_x'}{\sqrt{1-V^2/c^2}}\\[2ex] p_x=\dfrac{p_x'+\frac{V}{c^2}\varepsilon'}{\sqrt{1-\frac{V^2}{c^2}}}\\[2ex] \vec{p}_\perp=\vec{p}_\perp'\end{cases}$$

Transformation of the frequency ω of EM waves!

$\varepsilon=\hbar\omega$; $\vec{p}=\hbar\frac{\omega}{c}\hat{e}_k$



$$\hbar\omega'=\frac{\hbar\omega-\vec{V}\vec{p}}{\sqrt{1-\frac{V^2}{c^2}}}=\hbar\omega\,\frac{1-\frac{\vec{V}\cdot\hat{e}_k}{c}}{\sqrt{1-V^2/c^2}}$$

$$\boxed{\omega'=\omega\,\frac{1-\hat{e}_k\cdot\vec{V}/c}{\sqrt{1-V^2/c^2}}}$$

(a) $\hat{e}_k\parallel\vec{V}$ $(\hat{e}_k\equiv\hat{e}_x)$

$$\omega'=\omega\cdot\frac{1-\frac{V}{c}}{\sqrt{1-\frac{V^2}{c^2}}}=\omega\sqrt{\frac{1-\frac{V}{c}}{1+\frac{V}{c}}}$$

(b) $\hat{e}_k\perp\vec{V}$ $(\hat{e}_k\equiv\vec{e}_y)$

$$\omega'=\omega\Big/\sqrt{1-\frac{V^2}{c^2}}$$

Doppler effect:

"ω" depends on $\vec{V}$