Lecture 20   04/06/23

recap:

4D-momentum vector

$$\boxed{p^i = mcu^i = mc\frac{dx^i}{cd\tau} = \left(\underbrace{\frac{mc}{\sqrt{1-\frac{v^2}{c^2}}}}_{\text{Energy}/c}, \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right)}$$

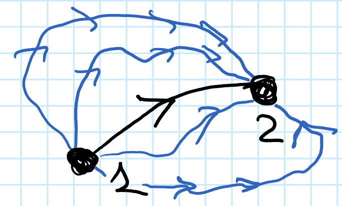
$$p^i p_i = m^2c^2 u^i u_i = mc^2$$

Relativistic energy: $\mathcal{E} = \dfrac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \boxed{\dfrac{\mathcal{E}^2}{c^2} - \bar{p}^2 = m^2c^2}$

Relativistic action: $S^{ac} = \int_1^2 L\,dt$

action

Lagrange function



The Least Action principle

S^{ac}_{12} = min. along the real trajectory

Classical Mechanics

$$L = \frac{m\dot{q}^2}{2} - U(q)$$

Kinetic energy   Potential energy

Relativistic Lagrange Function:

$S^{ac} = -mc \cdot s_{12}$ ← invariant interval

$$dS^{ac} = -mc\,ds = -mc^2\sqrt{1-\frac{v^2}{c^2}}\,dt$$

$$ds = c\,d\tau = c\,dt\sqrt{1-\frac{v^2}{c^2}}$$

proper time

Lagrange function:

$$\boxed{L = \frac{dS^{ac}}{dt} = -mc^2\sqrt{1-v^2/c^2}}$$

The total energy in the Classical Mechanics:

$L = \dfrac{m\dot{q}^2}{2} - U(q)$; $p = \dfrac{\partial L}{\partial \dot{q}}$; Energy $= \dot{q}\dfrac{\partial L}{\partial \dot{q}} - L$

$q = q(t)$ ← coordinate

$\dot{q} = v$ ← velocity

$(p = m\dot{q})$

$$\boxed{\mathcal{E} = \dot{q}\frac{\partial L}{\partial \dot{q}} - L}$$

$$\mathcal{E} = \dot{q}\frac{\partial L}{\partial \dot{q}} - L = \dot{q}\cdot m\dot{q} - \frac{m\dot{q}^2}{2} + U(q)$$

$$\mathcal{E} = \frac{m\dot{q}^2}{2} + U(q)$$

Relativistic energy:

$$\mathcal{E} = v\frac{\partial L}{\partial v} - L = -mc^2 \cdot v\frac{\partial}{\partial v}\left(\sqrt{1-\frac{v^2}{c^2}}\right) + mc^2\sqrt{1-\frac{v^2}{c^2}} =$$

$$= m\cancel{c^2}\frac{\cancel{2}v^2}{\cancel{2}\sqrt{1-\frac{v^2}{c^2}}\,\cancel{c^2}} + mc^2\sqrt{1-\frac{v^2}{c^2}} = \frac{\cancel{mv^2} + mc^2 - \cancel{mv^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\boxed{\mathcal{E} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}}$$

If $v \ll c$: $\mathcal{E} \simeq mc^2 + \dfrac{mv^2}{2} + \ldots$

The rest energy $\mathcal{E} = mc^2$ does not exist in Newton's Mechanics

4-invariant: $\boxed{p^i p_i = \left(\frac{\mathcal{E}}{c}\right)^2 - \vec{p}^{\,2} = \text{invariant}}$ ← square of the 4-momentum vector

There are different methods to calculate this invariant:

Method 1. $p^i \cdot p_i = \left(\frac{\mathcal{E}}{c}\right)^2 - \vec{p}^{\,2} = \frac{m^2c^2}{\left(1-\frac{v^2}{c^2}\right)} - \frac{m^2v^2}{\left(1-\frac{v^2}{c^2}\right)} = m^2c^2$

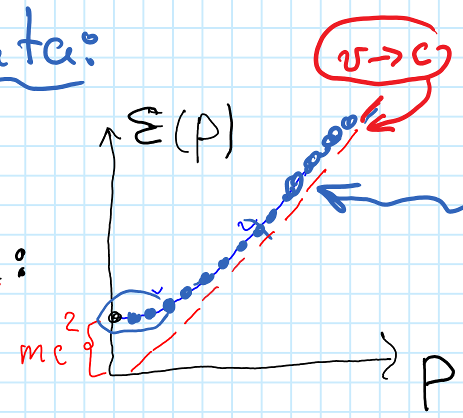
Method 2. In the proper frame: $v' = 0 \Rightarrow p^{i\prime} = (mc, 0)$

$$p^i p_i = p^{i\prime} p_i' = m^2c^2 \Leftarrow p^{i\prime} \cdot p_i' = (mc)^2 - 0^2 = m^2c^2$$

Method 3. $p^i = mcu^i \Rightarrow p^i \cdot p_i = m^2c^2\, u^i \cdot u_i = m^2c^2 \quad (u^i u_i = 1)$

Experimental Data:

Experimental check of the relativistic theory:



$\boxed{\frac{\mathcal{E}^2}{c^2} - \vec{p}^{\,2} = m^2c^2}$

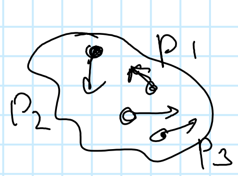
$\boxed{\mathcal{E} = \sqrt{m^2c^4 + p^2c^2}}$

Example: $v = c \Rightarrow \mathcal{E} = pc$ (photon) $(m_{photon} = 0)$ ← the rest mass of photon

Small v: $\mathcal{E} \simeq mc^2 + \frac{p^2}{2m}$

Large $v \to c$: $\mathcal{E} \simeq pc$

System of Particles:



$$p_\alpha^i = m_\alpha \frac{dx_\alpha^i}{ds} = \left(\frac{\mathcal{E}_\alpha}{c}, \vec{p}_\alpha\right) \qquad \vec{p}_\alpha = \frac{m_\alpha \vec{v}_\alpha}{\sqrt{1 - v_\alpha^2/c^2}}$$

↑ particle index

$$\begin{cases} p^i_{total} = \sum_\alpha p^i_\alpha = \left(\frac{\sum \mathcal{E}_\alpha}{c}, \sum_\alpha \vec{p}_\alpha\right) \\ p^i_{total} = \left(\frac{E_{total}}{c}, \vec{p}_{total}\right) \end{cases}$$

Momentum - Energy Conservation

$$\left.\begin{aligned} E_{total} &= \text{const} \\ \vec{p}_{total} &= \text{const} \end{aligned}\right\} \text{ for closed systems}$$

$$\boxed{\begin{aligned} E_{total} &= \sum_\alpha \mathcal{E}_\alpha = \sum_\alpha \frac{m_\alpha c^2}{\sqrt{1 - v_\alpha^2/c^2}} \\ \vec{p}_{total} &= \sum_\alpha \frac{m_\alpha \vec{v}_\alpha}{\sqrt{1 - v_\alpha^2/c^2}} \end{aligned}}$$

Lorentz transformations for E and $\vec{p}$:

Lorentz Transformation

$$\boxed{E = \frac{E' + V \cdot p_x'}{\sqrt{1 - \frac{V^2}{c^2}}}} \qquad \boxed{p_x = \frac{p_x' + \frac{V}{c^2} E'}{\sqrt{1 - \frac{V^2}{c^2}}}}$$

$$\boxed{p_y = p_y', \quad p_z = p_z'}$$

$$p^i = \begin{pmatrix} p_0 = \frac{mc}{\sqrt{1 - v^2/c^2}} \\ p_x = \frac{mv_x}{\sqrt{1 - \frac{v^2}{c^2}}} \\ p_y = \frac{mv_y}{\sqrt{1 - v^2/c^2}} \\ p_z = \frac{mv_z}{\sqrt{1 - \frac{v^2}{c^2}}} \end{pmatrix}$$

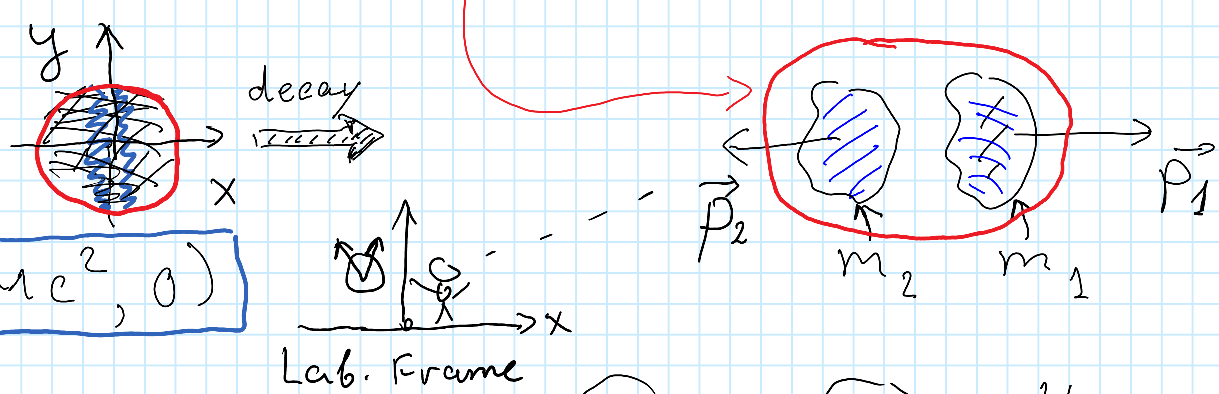
4D vector

Examples: Nuclear Fission

Unstable particle of Mass M decays into two particles

The proper frame for unstable particle:

$v = 0$



$$\boxed{P^i = (Mc^2, 0)}$$

4-vector of momentum

m_1 and m_2 — the rest masses of new particles

The particle mass is the same in all frames. It is an invariant

Calculate the energies of new particles ε_1 and ε_2.

Solution

For closed systems: $P^i_{initial} = const = P^i_{final}$ $\quad \boxed{\vec{p}_1 = -\vec{p}_2}$ momentum conservation

$$P^i_{final} = \left(\frac{\varepsilon_1 + \varepsilon_2}{c}, \vec{p}_1 + \vec{p}_2\right) = (Mc, 0)$$

Energy conservation

$$\begin{cases} \varepsilon_1 + \varepsilon_2 = Mc^2 \\ p_1 = p_2 \end{cases} \text{(vector form: } \vec{p}_1 = -\vec{p}_2\text{)} \Rightarrow p_1^2 = p_2^2$$

Momentum-energy conservation: $P^i_{initial} = P^i_{final}$

$$\frac{\varepsilon^2}{c^2} - p^2 = m^2c^2 \text{ invariant} \Rightarrow \frac{\varepsilon_1^2}{c^2} - m_1^2c^2 = \frac{\varepsilon_2^2}{c^2} - m_2^2c^2 \Rightarrow \varepsilon_1^2 - \varepsilon_2^2 = (m_1^2 - m_2^2)c^4$$

$$(\varepsilon_1 - \varepsilon_2)(\underbrace{\varepsilon_1 + \varepsilon_2}_{Mc^2}) = (m_1^2 - m_2^2)c^4$$

$$\begin{cases} \varepsilon_1 + \varepsilon_2 = Mc^2 \\ \varepsilon_1 - \varepsilon_2 = \dfrac{m_1^2 - m_2^2}{M}c^2 \end{cases}$$

$$\boxed{\begin{aligned} \varepsilon_1 &= \frac{M^2 + m_1^2 - m_2^2}{2M}c^2 \\ \varepsilon_2 &= \frac{M^2 + m_2^2 - m_1^2}{2M}c^2 \end{aligned}}$$

Simplest case: $m_1 = m_2 = m$

$$\varepsilon_1 = \varepsilon_2 = \varepsilon = \frac{M}{2}c^2$$

Definition of the kinetic energy:

$$K = \varepsilon(p) - mc^2 = mc^2\left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1\right)$$

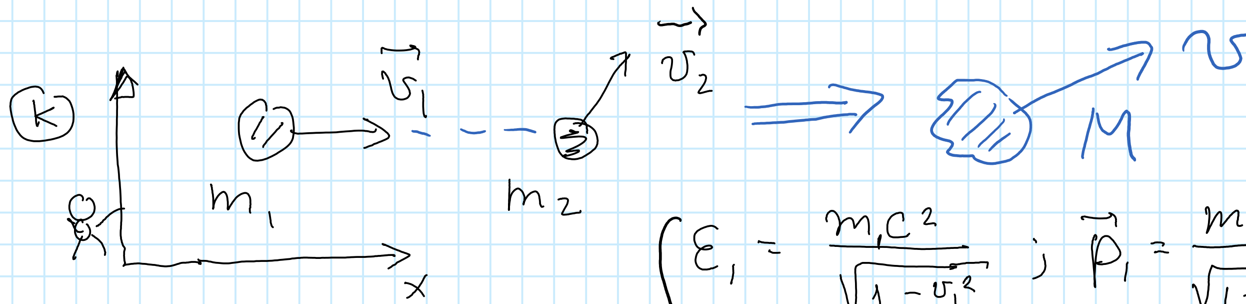
Particle kinetic energies:

$$K_1 = K_2 = K = \left(\frac{M}{2} - m\right)c^2 \Rightarrow \boxed{K_{total} = 2K = (M - 2m)c^2}$$

$(M > 2m)$

Example 2:

system of two particles



Inelastic collision

In (K) frame
$$\begin{cases} \mathcal{E}_1 = \dfrac{m_1 c^2}{\sqrt{1-\frac{v_1^2}{c^2}}} \; ; \; \vec{p}_1 = \dfrac{m_1 \vec{v}_1}{\sqrt{1-\frac{v_1^2}{c^2}}} \\ \mathcal{E}_2 = \dfrac{m_2 c^2}{\sqrt{1-\frac{v_2^2}{c^2}}} \; ; \; \vec{p}_2 = \dfrac{m_2 \vec{v}_2}{\sqrt{1-\frac{v_2^2}{c^2}}} \end{cases}$$

$$p^i_{\alpha=1} = \left(\frac{\mathcal{E}_1}{c}, \vec{p}_1\right)$$
$$p^i_{\alpha=2} = \left(\frac{\mathcal{E}_2}{c}, \vec{p}_2\right)$$

$$p^i_{total} = \left(\frac{\mathcal{E}_1 + \mathcal{E}_2}{c}, \vec{p}_1 + \vec{p}_2\right)$$

$$\boxed{p^i_{total} \cdot p_{i\,total} = \left(\frac{\mathcal{E}_1 + \mathcal{E}_2}{c}\right)^2 - (\vec{p}_1 + \vec{p}_2)^2 = = \text{invariant} \equiv M \cdot c^2}$$

The rest mass of a new particle M:

$$M^2 c^2 = \left(\frac{\mathcal{E}_1 + \mathcal{E}_2}{c}\right)^2 - (\vec{p}_1 + \vec{p}_2)^2 = \underbrace{\frac{\mathcal{E}_1^2}{c^2} - p_1^2}_{m_1^2 c^2} + \underbrace{\frac{\mathcal{E}_2^2}{c^2} - p_2^2}_{m_2^2 c^2} + 2\left(\frac{\mathcal{E}_1 \mathcal{E}_2}{c^2} - \vec{p}_1 \cdot \vec{p}_2\right)$$

The mass M of a new particle:

$$M^2 = \frac{1}{c^2}\left[m_1^2 c^2 + m_2^2 c^2 + 2 \frac{m_1 m_2 c^2 \left(1 - \frac{\vec{v}_1 \cdot \vec{v}_2}{c^2}\right)}{\sqrt{\left(1-\frac{v_1^2}{c^2}\right)} \cdot \sqrt{\left(1-\frac{v_2^2}{c^2}\right)}} \right]$$

$$M = \left[m_1^2 + m_2^2 + 2 m_1 m_2 + 2 m_1 m_2 \left(\frac{1 - \frac{\vec{v}_1 \vec{v}_2}{c^2}}{\sqrt{1-\frac{v_1^2}{c^2}} \cdot \sqrt{1-\frac{v_2^2}{c^2}}} - 1 \right) \right]^{1/2}$$

$$\boxed{M = \left[(m_1 + m_2)^2 + 2 m_1 m_2 \left(\frac{1 - \frac{\vec{v}_1 \cdot \vec{v}_2}{c^2}}{\sqrt{\left(1-\frac{v_1^2}{c^2}\right)\left(1-\frac{v_2^2}{c^2}\right)}} - 1 \right) \right]^{1/2}}$$

Simple case: $\vec{v}_2 = 0$;

$$\boxed{M = \left[(m_1 + m_2)^2 + 2 m_1 m_2 \left(\frac{1}{\sqrt{1-\frac{v_1^2}{c^2}}} - 1 \right) \right]^{1/2}}$$

$M \neq m_1 + m_2$

$M > m_1 + m_2$

Simplest case:

$\vec{v}_1 = \vec{v}_2 = \vec{v}$

$$M = \left[(m_1 + m_2)^2 + 2 m_1 \cdot m_2 \left(\cancel{\frac{1 - \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}} - 1} \right) \right] = m_1 + m_2$$

0