Lecture 19 04/03/23

recap: $u^i = dx^i/ds \leftarrow$ 4D velocity vector

$$\boxed{u^i = \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{\vec{v}}{c\sqrt{1-\frac{v^2}{c^2}}}\right)}$$

$$\boxed{u^i u_i = 1}$$

4D-vector of Acceleration

4D-velocity

$$u^i = \frac{dx^i}{ds} \quad \begin{matrix} \leftarrow \text{4D vector} \\ \leftarrow \text{4D scalar} \end{matrix}$$

$\Rightarrow$ 4D-vector of acceleration

$$w^i = \frac{du^i}{ds} \quad \begin{matrix} \leftarrow \text{vector} \\ \leftarrow \text{scalar} \end{matrix}$$

Relationship between u^i and w^i

4D-velocity

Invariant: $dx^i \cdot dx_i = ds^2 \Rightarrow \left(\frac{dx^i}{ds}\right)\left(\frac{dx_i}{ds}\right) = u^i \cdot u_i = 1$!!

$x^i = (ct, x, y, z)$; $ds^2 = (cdt)^2 - d\vec{r}^{\,2}$

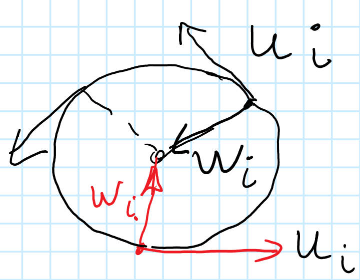
4-acceleration:

$$w^i = \frac{du^i}{ds} = \frac{d x^{i\,2}}{ds^2}$$

$$\boxed{u^i u_i = 1} \Rightarrow \underbrace{\frac{du^i}{ds} u_i + u^i \cdot \frac{du_i}{ds}}_{} = 0$$

normalized vector

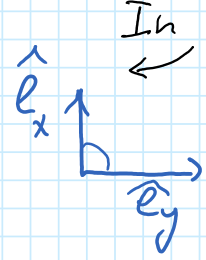
$$2 \cdot u_i \cdot \frac{du^i}{ds} = 0 \Rightarrow \boxed{u_i \cdot w^i = 0}$$



$$u_i \perp w_i$$

The formal analogy

In 4D-space



$\hat{e}_x \cdot \hat{e}_y = 0$

$\hat{e}_x \perp \hat{e}_y$

Example: Transformation of "4D"-velocities from K' to K frame

$$\begin{cases} u^0 = \dfrac{u^{0\prime} + \frac{V}{c} \cdot u^{1\prime}}{\sqrt{1-\frac{V^2}{c^2}}} \\ u^1 = \dfrac{\frac{V}{c} u^{0\prime} + u^{1\prime}}{\sqrt{1-\frac{V^2}{c^2}}} \\ u^2 = u^{2\prime} ; \; u^3 = u^{3\prime} \end{cases}$$

$u^{i\prime} = (u^{0\prime}, u^{1\prime}, u^{2\prime}, u^{3\prime})$ in the "4D"-space

$$u^{0\prime} = \frac{1}{\sqrt{1-\frac{v'^2}{c^2}}} ; \quad u^{1\prime} = \frac{v_x'}{c\sqrt{1-\frac{v'^2}{c^2}}} ;$$

$$u^{2\prime} = \frac{v_y'}{c\sqrt{1-\frac{v'^2}{c^2}}} ; \quad u^{3\prime} = \frac{v_z'}{c\sqrt{1-\frac{v'^2}{c^2}}}$$

Attention:
$\vec{V}$ - frame velocity
$\vec{v}$ - particle velocity

$$u^i u_i = 1 \Rightarrow u^{0\,2} - u^{1\,2} - u^{2\,2} - u^{3\,2} = \text{invar}$$

Home work 6

Problem 12.3 (c)

$v_A \le c$; $v_B \le c$

show that

$$v_{AB} = \frac{v_A + v_B}{1 + \frac{v_A v_B}{c^2}} \le c$$

Units: $c=1$. We have to show:

$$v_{AB} = \boxed{\frac{v_A + v_B}{1 + v_A \cdot v_B}} \le 1$$

$$\begin{cases} v_A \le 1 \\ v_B \le 1 \end{cases} \quad v_A + v_B \le 1 + v_A \cdot v_B$$

$$\hookrightarrow v_A(1-\cancel{v_B}) \le 1 - \cancel{v_B} \Rightarrow v_A \le 1 \Rightarrow v_{AB} \le c$$

Relativistic Momentum

Classical momentum: $\vec{p} = m\vec{v} = m\frac{dr}{dt}$

Classical kinetic energy: $\mathcal{E} = \frac{p^2}{2m} = \frac{mv^2}{2}$

Relativstic 4D-vector of Energy-Momentum

$$\boxed{p^i = mc\frac{dx^i}{ds} = mcu^i}$$

(4-vector of momentum)

vector $\begin{cases} dx^i = (cdt, d\vec{r}) \\ ds = cdt\sqrt{1-\frac{v^2}{c^2}} \end{cases}$ scalar

$$p^i = mc\left(\frac{cdt}{cdt\sqrt{1-\frac{v^2}{c^2}}}, \frac{d\vec{r}}{cdt\sqrt{1-\frac{v^2}{c^2}}}\right)$$

4-Vector

$$p^i = \left(\frac{mc}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right)$$

$$\begin{cases} p^0 = \frac{mc}{\sqrt{1-v^2/c^2}} \\ p^i = \frac{mv_i}{\sqrt{1-\frac{v^2}{c^2}}} \quad (i=1,2,3) \end{cases}$$

$$p^i = \left(\frac{\mathcal{E}}{c}, \vec{p}\right)$$

$$\boxed{\vec{p} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}} \leftarrow \vec{p}\text{ - relativistic momentum}$$

$$\boxed{\mathcal{E} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}} \leftarrow \mathcal{E}\text{ - relativistic energy}$$

Classical limits: $\boxed{v/c \ll 1}$

$$\vec{p} = m\vec{v}\left(1-\frac{v^2}{c^2}\right)^{-1/2} \simeq m\vec{v} + \frac{1}{2}m\vec{v}\frac{v^2}{c^2}$$

$$\mathcal{E} = mc^2\left(1-\frac{v^2}{c^2}\right)^{-1/2} \simeq mc^2 + \frac{mv^2\cancel{c^2}}{2\cancel{c^2}}$$

The energy at $v=0$: $\boxed{\mathcal{E}_0 = mc^2}$ ← the rest energy

This is the energy of the particle in the proper frame.