Lecture 18 03/31/23

4-Vectors

$\begin{cases} x^0 = ct; \\ x^1 = x \\ x^2 = y \\ x^3 = z \end{cases}$

Contravariant 4-vectors: $A^i = (A^0, A^1, A^2, A^3)$

transformations for different inertial frames $\rightarrow$
$$\begin{cases} A^0 = \dfrac{A^{0\prime} + \frac{V}{c} A^{1\prime}}{\sqrt{1 - V^2/c^2}} \\ A^1 = \dfrac{A^{1\prime} + \frac{v}{c} A^{0\prime}}{\sqrt{1 - \frac{V^2}{c^2}}} \\ A^2 = A^{2\prime}, \quad A^3 = A^{3\prime} \end{cases}$$

x-axis is oriented along $\vec{V}$

radius vector
(ct, x, y, z)
$(ct, \vec{r})$
$c^2t^2 - \vec{r}^{\,2} = \text{inv.}$

4-vectors:
$A^i = (A^0, \vec{A})$ $\quad \vec{A} \equiv (A^1, A^2, A^3)$ $\quad A_x, A_y, A_z$

A^0: time-like component

$\vec{A}$: space-like component, 3D-vector

Example: 4-radius vector $\quad x^i = (ct, x, y, z) = (x^0, x^1, x^2, x^3)$

Covariant vectors:

$A_i = (A_0, A_1, A_2, A_3) = (A^0, -\vec{A})$

example: $x_i = (ct, -x, -y, -z) = (ct, -\vec{r})$

$\Rightarrow A^0 = A_0; \; A^1 = -A_1; \; A^2 = -A_2; \; A^3 = -A_3$

Relativistic invariants (Lorentz invariant) $\quad$ $ds \equiv \text{scalar}$

Scalar or invariant $\rightarrow$ $\boxed{d\tau = dt' \sqrt{1 - \frac{v^2}{c^2}}}$ $\leftarrow$ The proper time $d\tau$ has the same value in all moving frames

Scalar product of 4-vectors:

$\boxed{A^i \cdot B_i = \sum_{i=0}^{3} A^i B_i = A^0 \cdot B_0 - \vec{A} \cdot \vec{B}}$ $\leftarrow$ example (interval)

$A^i = x^i = (ct, \vec{r}); \; B_i = x_i = (ct, -\vec{r})$

$x^i \cdot x_i = (ct)^2 - \vec{r}^{\,2} = \text{invariant}$

4-velocity vector

$dx^i = (c\,dt, d\vec{r}) \leftarrow$ 4-vector of displacement

$ds = \sqrt{ds^2} = c\,dt \sqrt{1 - \frac{v^2}{c^2}} \in$ 4-scalar (invariant)

particle velocity $\boxed{\vec{v} = \frac{d\vec{r}}{dt}}$

Definition of 4-velocity: $u^i = \frac{dx^i}{ds} = \frac{dx^i}{c\,d\tau} \rightarrow$

$\boxed{u^i = \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \frac{\vec{v}}{c\sqrt{1 - \frac{v^2}{c^2}}} \right)}$

$[u^i]$ - dimensionless velocity

$\boxed{u^i u_i = \frac{1}{\left(1 - \frac{v^2}{c^2}\right)} - \frac{v^2}{c^2\left(1 - \frac{v^2}{c^2}\right)} = 1}$

The dimensional 4-velocity $\rightarrow$ $\boxed{c \cdot u^i = \left(\frac{c}{\sqrt{1 - \frac{v^2}{c^2}}}, \frac{\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \right)}$