Lecture 15

03/21/2023

We have studied:
- classical sources of radiation ($v \ll c$)
- classical interaction between particles and the EM field ($v \ll c$)

What should be observed, if $v \sim c$?

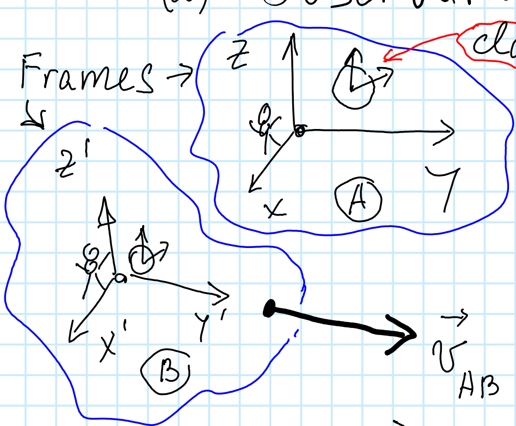
The Relativistic Theory

(a) Observations of physical events

Reference frame

System of coordinates + Clock

Frames →



Independent Observers

Inertial Frames: objects have to move with a constant velocity, if $\sum_i \vec{F}_i = 0$

$\vec{v}_{AB}$

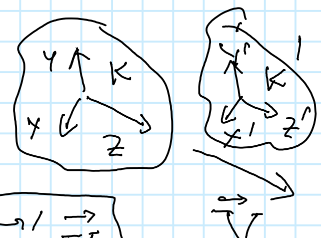
Inertial frames may have different relative velocities.

$\vec{v}_{AB} = \overrightarrow{const}$

Principle of Relativity: "Laws of nature have to be identical in all inertial reference frames"

Equations, describing these laws, must be invariant with respect to transformations of coordinates and time from one inertial frame to another.

Galilean Transformation and Maxwell's wave equation → $\begin{cases} x = x' + V \cdot t' \\ t = t' \end{cases}$



$\vec{V}$

$\boxed{\vec{v} = \vec{v}\,' + \vec{V}}$

EM-Wave equation for $f(x,t)$ function:

"f" could be an electric or magnetic field

"c" – the speed of light

$$\boxed{\frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}} \Rightarrow$$

Solution of the EM-wave equation

$$\boxed{f(x,t) = f_+(x - ct) + f_-(x + ct)}$$

This is the solution of Maxwell's wave equation in the K-frame.

recap: $\begin{cases} \eta = x - ct \\ \xi = x + ct \end{cases} \Rightarrow \frac{\partial^2 f}{\partial \xi \cdot \partial \eta} = 0 \Rightarrow \boxed{f(x,t) = f_+(\eta) + f_-(\xi)}$

The general solution

How would these solutions look in the K'-frame, if we apply Galilean Transformation: $\begin{cases} x = x' + Vt' \\ t = t' \end{cases}$

$$\begin{cases} \eta = x - ct = x' + Vt' - ct' = x' - (c - V)t' \\ f(x,t) = f_+[x - ct] + f_-[x + ct] = f_+[x' - (c-V)t'] + f_-[x' + (c+V)t'] \end{cases}$$

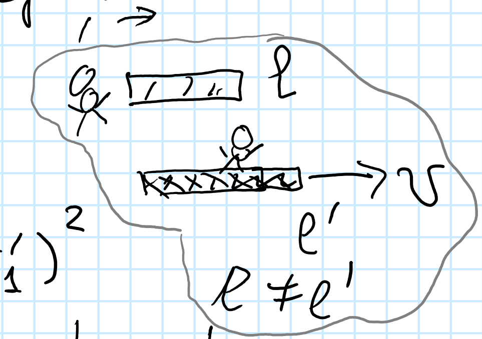
These solutions show, that the speed of propagation of EM waves depends on the velocity of frame motion (V) and corresponds to Galilean Law for velocity transformation: $\boxed{c' = c \pm V}$

? ?

The experimental data show, that the speed c of EM waves is the same in all inertial frames. We have to reject Galilean transformation:

$$\boxed{x \neq x' + Vt'},\ \boxed{t \neq t'} \text{ and } \boxed{\vec{v} \neq \vec{v}' + \vec{V}}.$$

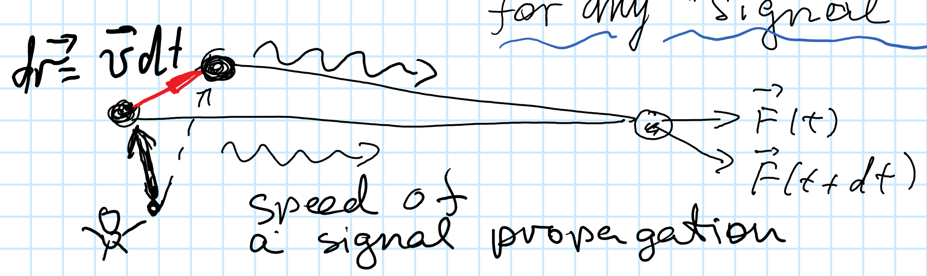
$$\text{and} \Rightarrow (x_2 - x_1)^2 \neq (x_2' - x_1')^2$$



Experiments:

c – maximal velocity of propagation of the EM interaction

It was shown in many experiments with charged and neutral particles that "c", the light speed in vacuum, is maximal speed for any "signal" propagation.



(time delay $\tau = \frac{|\vec{r} - \vec{r}'|}{c}$)

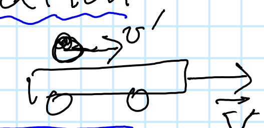
Classical Mechanics: (a) time is absolute $\quad t = t'$

(б) distance between points is absolute $\quad x_2 - x_1 = x_2' - x_1'$

Galilean Transformation

Maxwell Equations are not invariant with respect to Galilean Transformation

$$\begin{cases} x = x' + Vt' \\ t = t' \end{cases}$$



$$\vec{v} = \vec{v}' + \vec{V}$$

$$f(x - ct) = f(x' - (c - v)t')$$

Relativistic Intervals

All events are described by "place" and "time".

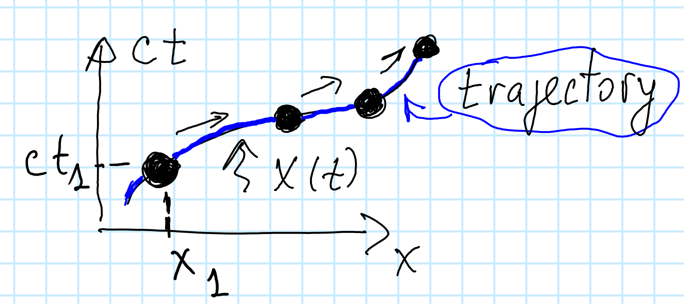
It is convenient to use a fictitious 4D space:

x, y, z and t

World point

World line

$\{ct, x(t), y(t), z(t)\}$

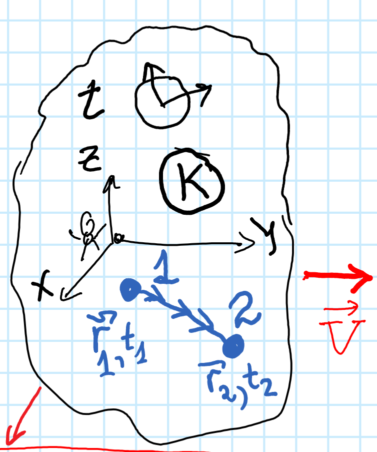


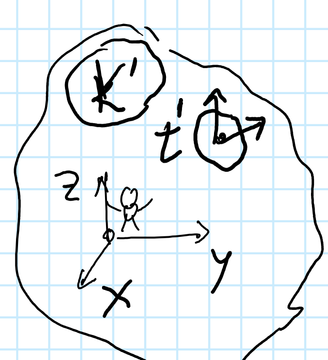
Light propagation: in K frame:

$\{ct_1, \vec{r}_1\}$ and $\{ct_2, \vec{r}_2\}$

source

detector





In K' frame:

source: $\{ct_1', \vec{r}_1'\}$

detector: $\{ct_2', \vec{r}_2'\}$

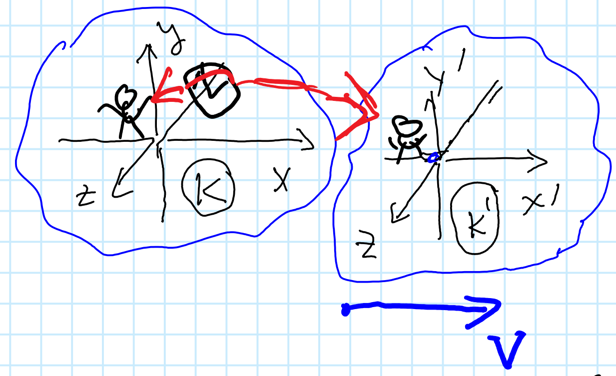
$$c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 = 0$$

K-frame

$$c^2(t_2' - t_1')^2 - (x_2' - x_1')^2 - (y_2' - y_1')^2 - (z_2' - z_1')^2 = 0$$

$$|\vec{r}_2 - \vec{r}_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

New transformation $(t,x)\to(t',x')$ must be used instead Galilean:



$$\begin{cases} x = \dfrac{x' + Vt'}{\sqrt{1-\frac{V^2}{c^2}}} \\ t = \dfrac{t' + \frac{V}{c^2}x'}{\sqrt{1-\frac{V^2}{c}}} \\ z = z',\ y = y' \end{cases}$$

$$\vec{V} = V\cdot\hat{e}_x$$

$$\boxed{\text{Lorentz Transformations } (t,\vec{r}) \leftrightarrow (t',\vec{r}\,')}$$ ← fomulated in advance

We can derive Lorentz transformation using relativistic intervals.

Invariant of the Lorentz transformations:

$$\boxed{S_{21}^2 = c^2(t_2-t_1)^2 - |\vec{r}_2-\vec{r}_1|^2 = \text{invariant}}$$

This is equivalent to Einstein's postulate $\boxed{c = \text{const}}$

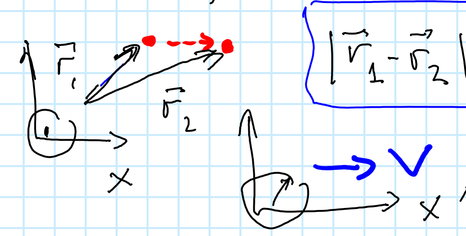
For EM waves: $S_{21}^2 = c^2\Delta t^2 - \Delta x^2 = 0$

$\Rightarrow \Delta x = c\cdot\Delta t$

Galilean transformation:

$$\begin{cases} x = x' + vt' \\ t = t' \\ z = z';\ y = y' \end{cases}$$

Invariant of Galilean transformations



$$\boxed{|\vec{r}_1 - \vec{r}_2|^2 = \text{invarint}}$$

moving frame

Lecture 16

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Math. Appendix (Derivation of Lorentz transformation)

We can derive the Lorentz formules using the postulate $S_{12}^2 = \text{invariant}$. To simplify formulas we select $t_1 = 0, x_1 = 0, y_1 = 0, z_1 = 0$

$$S^2 = ct^2 - x^2 - y^2 - z^2 = \text{invariant}$$

Transformation:

(Non-relativistic limit: $x = x' + Vt'$)

$$\begin{cases} x = \alpha_{11}x' + \alpha_{12}ct' \\ ct = \alpha_{21}x' + \alpha_{22}ct' \end{cases}$$

The limit of small velocities $v \ll c$: all formulas have to be identical to Galilean transformation.

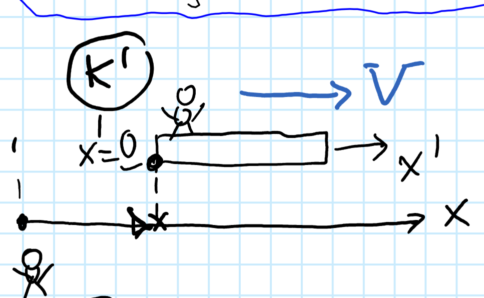
$$\begin{cases} x = x'\cosh\psi + ct'\cdot\sinh\psi \\ ct = x'\cdot\sinh\psi + ct'\cosh\psi \end{cases}$$

$$c^2t^2 - x^2 = c^2t'^2 - x'^2$$

$\cosh^2\psi - \sinh^2\psi = 1$ "1"

$$c^2t^2 - x^2 = c^2t'^2(\cosh^2\psi - \sinh^2\psi) - x'^2(\cosh^2\psi - \sinh^2\psi) +$$
$$+ 2x'ct'(\sinh\psi\cdot\cosh\psi - \sinh\psi\cosh\psi) =$$
$$= c^2t' - x'^2 = \text{invariant}$$

$\psi = ?$



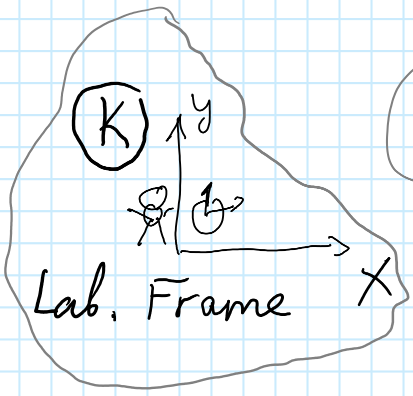
$$\frac{1}{\cosh^2\psi} = 1 - \tanh^2\psi$$

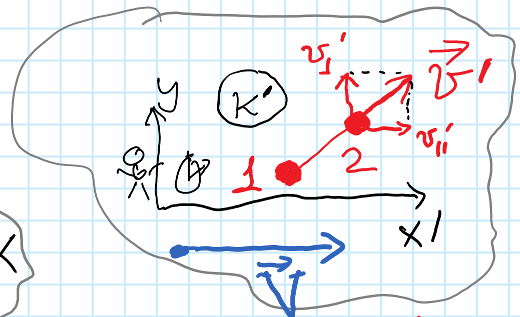
For $x' = 0$ → $\begin{cases} x = ct'\cdot\sinh\psi \\ ct = ct'\cosh\psi \end{cases} \Rightarrow \boxed{\tanh\psi = \frac{x}{ct} = \frac{V}{c}}$

$x/t = v$ ← the velocity of the K' frame

$$\boxed{\cosh\psi = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \qquad \sinh\psi = \frac{\frac{V}{c}}{\sqrt{1-\frac{V^2}{c^2}}}}$$

Transformation of Velocities





$\vec{v}_{\parallel} = v_x' \cdot \hat{e}_x$

$$\begin{cases} dx = \dfrac{dx' + V\cdot dt'}{\sqrt{1-\frac{V^2}{c^2}}} = dt'\,\dfrac{v_x' + V}{\sqrt{1-\frac{V^2}{c^2}}}\,; \\[2ex] dt = \dfrac{dt' + \frac{V}{c^2}\cdot dx'}{\sqrt{1-\frac{V^2}{c^2}}} = dt'\,\dfrac{1 + v_x' V/c^2}{\sqrt{1-\frac{V^2}{c^2}}}\,; \\[2ex] dy = dy'\,; \quad dz = dz' \end{cases}$$

$dx' = v_x' \cdot dt'$

$v_x' = \dfrac{dx'}{dt'}$

Particle velocities:

K' - frame: $\vec{v}\,' = v_x' \hat{e}_x' + v_y' \hat{e}_y' + \hat{v}_z' \hat{e}_z$

K - frame: $\vec{v} = v_x \hat{e}_x + v_y \hat{e}_y + v_z \hat{e}_z$

$\hat{e}_x = \hat{e}_x'\,; \quad \hat{e}_y = \hat{e}_y\,; \quad \hat{e}_z = \hat{e}_z'$

$\vec{V} = \hat{e}_x V$

K- frame velocity

$$\boxed{v_x = \frac{dx}{dt} = \frac{v_x' + V}{1 + \frac{v_x' V}{c^2}}}\,; \qquad \boxed{v_y = \frac{v_y' \sqrt{1-\frac{V^2}{c^2}}}{1 + \frac{v_x' V}{c^2}}}\,; \qquad \boxed{v_z = \frac{v_z' \sqrt{1-\frac{V^2}{c^2}}}{1 + \frac{v_x' V}{c^2}}}$$

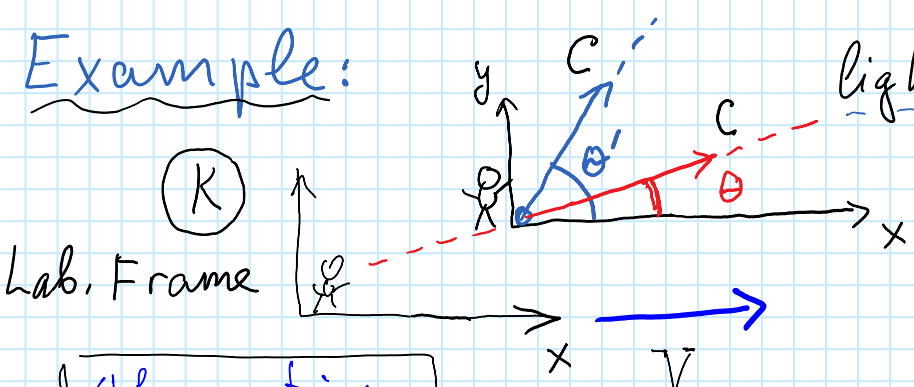
The same formulas expressed in the vector form:

$$\boxed{\vec{v}_{\parallel} = \frac{\vec{v}_{\parallel}\,' + \vec{V}}{1 + \frac{\vec{v}\,'\vec{V}}{c^2}}} \qquad \boxed{\vec{v}_{\perp} = \vec{v}_{\perp}\,' \cdot \frac{\sqrt{1 - V^2/c^2}}{1 + \frac{\vec{v}\,'\cdot\vec{V}}{c^2}}}$$

$\vec{v}_{\parallel}\,' \parallel \vec{V}$

$\vec{v}_{\perp}\,' \perp \vec{V}$

Example: light propagation in K and K' frames



Aberration of light

$v = v' = c$

$$\boxed{\cos\theta = \frac{\cos\theta' + \frac{V}{c}}{1 + \frac{V}{c}\cos\theta'}} \qquad \theta' \to \theta$$

$(v_x' = c\cdot\cos\theta'\,; \quad v_y' = c\cdot\sin\theta')$

Transformation $\theta \to \theta'$:

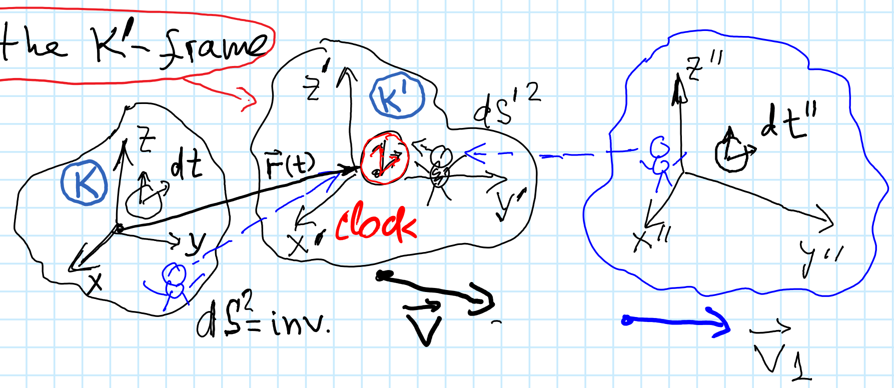
$V \to -V$

$$\boxed{\cos\theta' = \frac{\cos\theta - \frac{V}{c}}{1 - \frac{V}{c}\cos\theta}}$$

Proper Time

Clock is placed into the K′-frame

$\boxed{ds^2 = ds'^2 = ds''^2 = \ldots}$



For the K-frame:

$$ds = \sqrt{c^2dt^2 - d\vec{r}^{\,2}} =$$

$$= c\,dt\sqrt{1 - \frac{1}{c^2}\left(\frac{d\vec{r}}{dt}\right)^2}$$

This is the clock velocity for observer in the K-frame

$\vec{v} = \vec{V}$ for K-observer.

$$ds = c\,dt\sqrt{1 - \frac{v^2}{c^2}}$$

("v" is the relative velocity of "moving" clocks and observer K′)

$$ds'' = c\,dt''\sqrt{1 - \frac{v''^2}{c^2}}$$

In the specific frame K′: $v = 0$ (Clock is not moving in the frame associated with the clocks.)

$v_{clock} = V_{frame}$

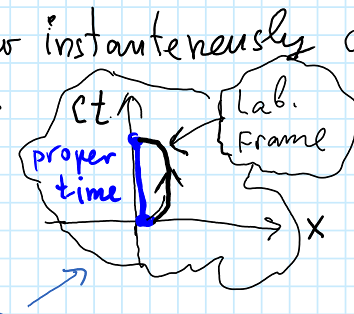
$ds' = c\,dt'$ → $\boxed{dS' \equiv c\,d\tau = c\,dt\sqrt{1 - \frac{v^2}{c^2}}}$ ⇒ $\boxed{\tau = (t_2 - t_1)\sqrt{1 - \frac{v^2}{c^2}}}$ ← **Proper time** τ

the clock time-interval, if $d\vec{r} \equiv 0$

If, $\vec{V} \equiv \vec{v}(t)$

$$\boxed{\tau \le t_2 - t_1} \leftarrow \boxed{\tau = \int_{t_1}^{t_2} dt\sqrt{1 - \frac{\vec{v}(t)^2}{c^2}}}$$

Observer instanteneusly changes frames



$$\boxed{S_{12} = \int_{(1)}^{(2)} dS \equiv S'_{12}}$$

invarient for inertial frames

(1) and (2) are some events

The time read by a clock moving with a given object is called the proper time for this object.

$$t_2 - t_1 = \int_{t_1}^{t_2} dt$$ → is the time-interval in the reference frame, where the motion of the clock is observed.

$$\boxed{\tau \le (t_2 - t_1)}$$

Physical Example:

Life-time of moving unstable particles: μ^- – muon

$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$ ← neutrino

$\boxed{\tau_L \cong 2.2 \cdot 10^{-6}\,s}$ proper life-time

for $v \to c$

$$\boxed{t_\ell = \frac{\tau_L}{\sqrt{1 - v^2/c^2}} \gg \tau_L}$$

Laboratory life-time