Lecture 14

03/07/2023

recap:

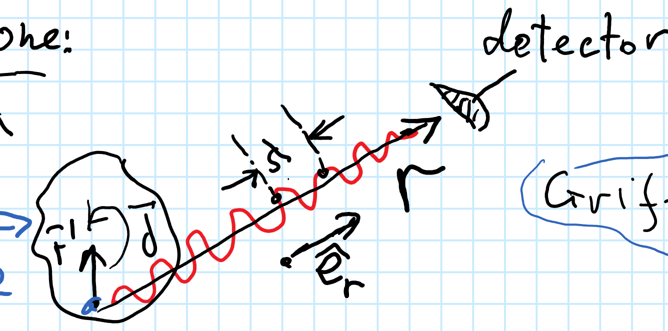
Solution of wave equation

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,', t - \frac{|\vec{r}-\vec{r}\,'|}{c})}{|\vec{r}-\vec{r}\,'|}\, d^3r'}$$

↖ vector-potential

radiation zone:

$r \gg \vec{r}\,', \lambda$



Griffith 11.1

$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0\, e^{i(kr-\omega t)}}{4\pi r}\int \vec{J}_\omega(\vec{r}\,')\cdot e^{-i\vec{k}\vec{r}\,'}\, d^3r'}$$

Simplest case:

$$\begin{cases} \vec{J}(\vec{r},t) = \vec{J}_\omega(\vec{r}\,')\, e^{-i\omega t} \\ \rho(\vec{r},t) = \rho_\omega(\vec{r}\,')\cdot e^{-i\omega t} \end{cases}$$

$\lambda = cT = \frac{2\pi}{k}$ - wave length

$T = 2\pi/\omega$ - period

Harmonic sources

$$\boxed{\begin{aligned} k &= \frac{\omega}{c} \\ \vec{k} &= k\cdot\hat{e}_r \end{aligned}}$$

$\hat{e}_r \equiv \hat{e}_k$

$k = \frac{2\pi}{\lambda}$

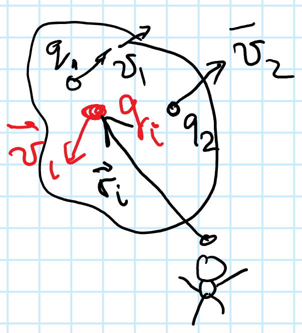
Wave vector $\vec{k}$

if $kr' = 2\pi\frac{r'}{\lambda} \ll 1$

We can neglect all small terms in the $\vec{A}$-formula

$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\,\frac{e^{i(\vec{k}\vec{r}-\omega t)}}{r}\cdot\int \vec{J}_\omega(\vec{r}\,')\, d^3r'}$$

General formula for calculations $\int_V \vec{J}(\vec{r}\,',t)\, d^3r'$



We can consider a system of point charges $\{q_i\}$

$\rho(\vec{r},t) = \sum_i q_i\,\delta(\vec{r}-\vec{r}_i(t))$ ← The volume charge density

$\vec{v}_i = \frac{d\vec{r}_i(t)}{dt}$

$$\int \vec{J}(\vec{r}_i,t)\, d^3r' = \int \sum_i q_i\,\delta(\vec{r}\,'-\vec{r}_i(t))\cdot\vec{v}_i\, d^3r' =$$

$$= \sum_i q_i \int \vec{v}_i\,\delta(\vec{r}\,'-\vec{r}_i(t))\, d^3\vec{r}\,' = \sum_i q_i\,\vec{v}_i = \sum_i q_i\frac{d\vec{r}_i}{dt} = \frac{d}{dt}\Big(\sum_i q_i\vec{r}_i\Big)$$

$\vec{r}\,' \to \vec{r}_i(t)$

The definition of a dipole moment: $\vec{p} = \int \rho(\vec{r},t)\,\vec{r}\, d^3r = \sum_i q_i\,\vec{r}_i(t)$

(for the system of point charges)

$$\boxed{\int \vec{J}(\vec{r},t)\, d^3r = \frac{d}{dt}\vec{p}(t) = \dot{\vec{p}}(t)}$$

where $\vec{p}(t)$ is the system dipole moment.

The vector-potential in radiation zone for the oscillating dipole $\vec{p}(t)$:

$\vec{J}(\vec{r},t) = \vec{J}_\omega(\vec{r})\cdot e^{-i\omega t}$

$\vec{p}(t) = \vec{p}_\omega\cdot e^{-i\omega t}$

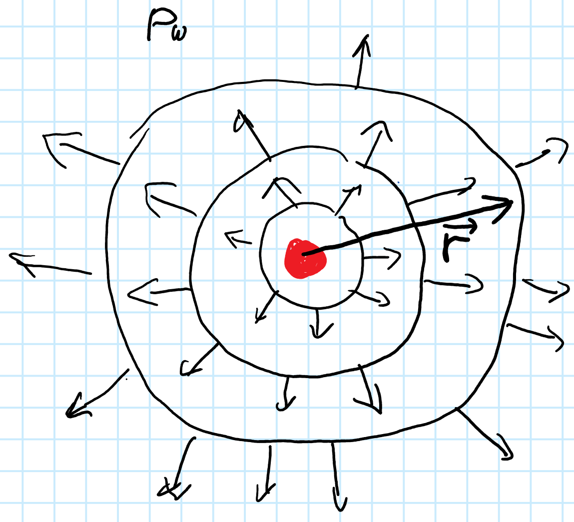
$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\,\frac{e^{i(kr-\omega t)}}{r}\cdot(-i\omega)\,\vec{p}_\omega = -i\frac{\mu_0\omega}{4\pi}\cdot\frac{e^{i(kr-\omega t)}}{r}\cdot\vec{p}_\omega}$$

$$\int \vec{J}(\vec{r},t)\, d^3r = \frac{d}{dt}\Big(\int\rho(\vec{r},t)\,\vec{r}\Big) = -i\omega e^{-i\omega t}\underbrace{\int \rho_\omega(\vec{r}\,')\,\vec{r}\, d^3r'}_{\vec{p}_\omega} = -i\omega e^{-i\omega t}\cdot\vec{p}_\omega$$

$\dot{\vec{p}}(t) = -i\omega\,\vec{p}(t)$

Spherical wave:

radiation zone $r \gg \lambda, r'$



phase:

$$\boxed{\begin{aligned} \eta &= kr - \omega t = \\ &= k(r - ct) \end{aligned}}$$

$\omega = ck$

The vector potential, induced by harmonically oscillated electric dipole

$$E(\vec r,t)=? \quad B(\vec r,t)=?$$

$$\vec A(\vec r,t)=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\dot{\vec p}_\omega(t)=-\frac{\mu_0 i\omega}{4\pi}\frac{e^{i(kr-\omega t)}}{r}\cdot\vec p_\omega$$

creates in the radiation zone the $\vec B$ and $\vec E$ fields.

Magnetic field in the far (radiation) zone $\boxed{r\gg r',\lambda}$:

$\lambda=\frac{2\pi}{k}$

Wave vector:

$\vec k=k\cdot\hat e_r$

$\hat k=\frac{\vec r}{r}=\hat e_r$

$\hat e_k\equiv\hat k\equiv\hat e_r$

$$\vec B=\vec\nabla_r\times\vec A=\frac{\mu_0}{4\pi}\vec\nabla_r\times\frac{e^{ikr}}{r}\cdot\dot{\vec p}_\omega(t)=\frac{\mu_0}{4\pi}\vec\nabla\left(\frac{e^{ik\bar r}}{r}\right)\times\dot{\vec p}_\omega(t)=$$

$$=\frac{\mu_0}{4\pi}\left[\left(\frac{i\vec k}{r}-\cancel{\frac{\vec r}{r^3}}\right)e^{ik\bar r}\times\dot{\vec p}_\omega(t)\right]=\frac{\mu_0}{4\pi}\frac{e^{ik\bar r}}{r}\cdot\left[i\vec k\times\dot{\vec p}_\omega(t)\right]$$

Terms: Large $k\sim\frac{1}{\lambda}$ small $r\gg\lambda$

$\vec\nabla_r\cong i\vec k$

$\dot{\vec p}=-i\omega\vec p_\omega(t)$

$k=\omega/c$

Harmonic Sources

$\vec p(t)=\vec p_\omega\cdot e^{-i\omega t}$

$$\boxed{\vec B=\frac{\mu_0\omega^2}{4\pi c}(\hat k\times\vec p_\omega)\cdot\frac{e^{i(kr-\omega t)}}{r}}$$

"complex numbers"

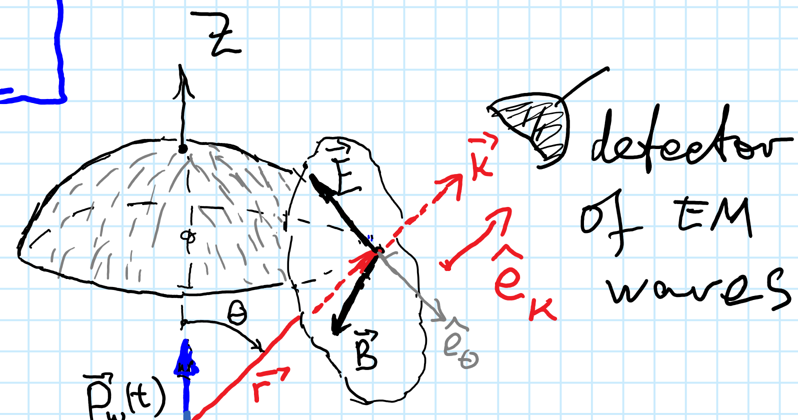
$\hat k\equiv\hat e_r=\frac{\vec r}{r}$

$\vec p_\omega\uparrow\uparrow\hat e_z$

$\hat k\times\vec p_\omega=-\hat e_\varphi\sin\theta$

$\vec B\perp\vec k$ and $\vec B\perp\vec p$

for $\vec p(t)=\vec p_\omega\cdot e^{-i\omega t}$



$$\boxed{\vec B=-\vec e_\varphi\,\frac{\mu_0\omega^2}{4\pi c}p_\omega\cdot\sin\theta\,\frac{\cos(kr-\omega t)}{r}}$$

"real part" in the spherical coordinates.

$\ddot{\vec p}_\omega=-\omega^2\cdot\vec p_\omega(t)$

$$\boxed{\vec B=\frac{\mu_0}{4\pi c}\frac{e^{ikr}}{r}\left(\hat k\times\ddot{\vec p}_\omega(t)\right)}$$

The electric field can be calculate, using our general formula for the EM waves:

Maxwell's eq. HW

$\vec\nabla\times\vec B=\mu_0\varepsilon_0\frac{\partial\vec E}{\partial t}$

$i\vec k\times\vec B=\frac{1}{c^2}(-i\omega)\vec E$

$\vec E\perp\vec B,\hat k$

$$\boxed{\vec E=c\,(\vec B\times\hat k)}$$

$$\boxed{\vec E=-\hat e_\theta\,\frac{\mu_0\omega^2\cdot p_\omega\cdot\sin\theta}{4\pi}\cdot\frac{\cos(kr-\omega t)}{r}}$$

complex value

$$\boxed{\vec E=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left[\hat k\times(\hat k\times\ddot{\vec p}_\omega(t))\right]}$$

Poynting vector.

$\vec p=\vec p_\omega\cdot e^{-i\omega t}$

$$\vec S=\frac{\vec E\times\vec B}{\mu_0}=\hat k\,\frac{\mu_0\,\omega^4\,p_\omega^2\cdot\sin^2\theta}{4\pi\cdot4\pi r^2c}\cdot\cos^2(kr-\omega t)$$

It can be written as:

$$\boxed{\vec S(\vec r,t)=\hat k\cdot\frac{\mu_0}{4\pi c}\,\frac{|\ddot{\vec p}_\omega(t)|^2\cdot\sin^2\theta\cdot\cos^2(kr-\omega t)}{4\pi r^2}}$$

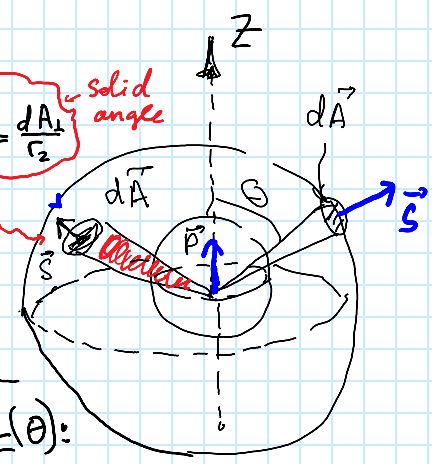
$\ddot{\vec p}=-\omega^2\cdot\vec p_\omega(t)$

Power of the emitted EM radiation:

$$dP = \vec{S}\cdot d\vec{a} = S\cdot da_{\perp} = \frac{\mu_0 p_\omega^2 \cdot \omega^4 \cdot \sin^2\theta \cos^2(kr-\omega t)}{4\pi c \; 4\pi r^2}\, da_{\perp}$$

energy power through dA

$$d\Omega = \frac{dA_\perp}{r^2} \text{ solid angle}$$



Angular intensity of emission

$$\boxed{I(\theta) = \left(\frac{dP}{d\Omega}\right) = \frac{\mu_0 p_\omega^2\, \omega^4}{16\pi^2 c}\sin^2\theta\, \cos^2(kr-\omega t)}$$

$$\boxed{\langle I(\theta)\rangle_t = \frac{\mu_0 p_\omega^2 \omega^4}{32\pi^2 c}\sin^2\theta}$$

The averaged over time $I(\theta)$:

$$\langle \cos^2(kr-\omega t)\rangle_t = 1/2$$

The total power of radiation:

$$d\Omega = 2\pi \sin\theta\, d\theta$$

$$\langle P(t)\rangle_t = \oint_A \langle\vec{S}\rangle_t\, d\vec{a} = \int_{4\pi}\langle I(\theta)\rangle_t \cdot d\Omega = \frac{\mu_0 p_\omega^2 \omega^4}{32\pi^2 c}\int_0^\pi \sin^2\theta\cdot 2\pi\sin\theta\, d\theta = \frac{\mu_0 p_\omega^2\omega^4}{16\pi^2 c}\cdot\frac{4}{3}$$

area

$$\int_0^\pi \sin^2\theta \sin\theta\, d\theta = \int_{-1}^{1}(1-x^2)\,dx = \left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1} = \frac{4}{3}$$

$x = \cos\theta$

$$\boxed{\langle P\rangle_t = \frac{\mu_0 p_\omega^2\cdot\omega^4}{4\pi\cdot 3c}} \rightarrow \boxed{\langle P\rangle_t = \frac{1}{4\pi\varepsilon_0}\frac{p_\omega^2\cdot\omega^4}{3c^3}}$$

$$\mu_0\cdot\varepsilon_0 = 1/c^2$$

In Gaussian units: $\frac{1}{4\pi\varepsilon_0} \equiv 1$

$$\boxed{\langle P\rangle_t = \frac{p_\omega^2\,\omega^4}{3c^3}}$$

The same formula can be written using an expression for the oscillating dipole $\vec{p}_\omega(t) = \vec{p}_0 e^{-i\omega t}$:

$$\text{Eq.}(*) \quad \boxed{\langle \vec{P}\rangle_t = \frac{\mu_0}{4\pi}\frac{|\ddot{\vec{p}}_\omega(t)|^2}{3c} = \frac{1}{4\pi\varepsilon_0}\frac{|\ddot{\vec{p}}_\omega(t)|^2}{3c^3}}$$

where $\ddot{\vec{p}}_\omega(t) = -\omega^2\vec{p}_\omega(t)$.

The time-dependent power of emitted EM wave can be written as Eq.(*) for any dependence $\vec{p}(t)$:

$$\boxed{P(t) = \frac{\mu_0}{4\pi}\,\frac{2}{3}\,\frac{|\ddot{\vec{p}}(t_r)|^2}{c}}$$

$\langle\cos^2\eta\rangle_t = 1/2$

$\eta = kr - \omega t$

for harmonic waves

$$t_r = t - \frac{r}{c} \Leftarrow \text{the time delay}$$

$$\tau_d = \frac{r}{c}$$

Dipole radiation of EM wave:

$$I(\theta) = \frac{dP}{d\Omega} \propto \sin^2\theta$$