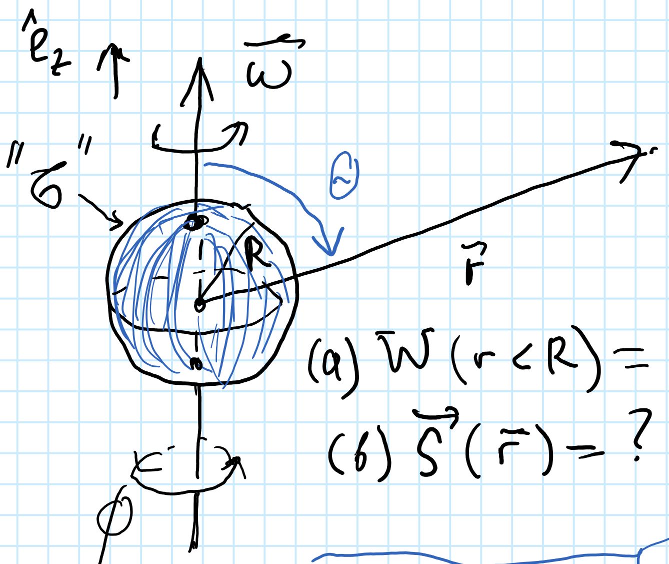
Lecture 13 — 03/02/2023

Problem 4 — Solution of HW Problems:



(a) $W(r<R) = ?$

(b) $\vec{S}(\vec{r}) = ?$

surface charge density $\rightarrow$ $\boxed{\sigma = q/4\pi R^2}$

From the Griffiths example 5.11: The vector potential $\vec{A}(\vec{r})$

Hint from Griffiths' textbook:

$r \le R$:
$$\begin{cases} \vec{A}(\vec{r}) = \dfrac{\mu_0 \cdot R \cdot \omega \sigma}{3} \cdot r \cdot \sin\theta \cdot \hat{e}_\phi \\ \vec{B}(\vec{r}) = \dfrac{2}{3}\mu_0 \sigma R \vec{\omega} \end{cases}$$

$r \ge R$:
$$\vec{A}(\vec{r}) = \frac{\mu_0 R^4 \omega \sigma}{3 \cdot r^2} \cdot \sin\theta \cdot \hat{e}_\phi$$

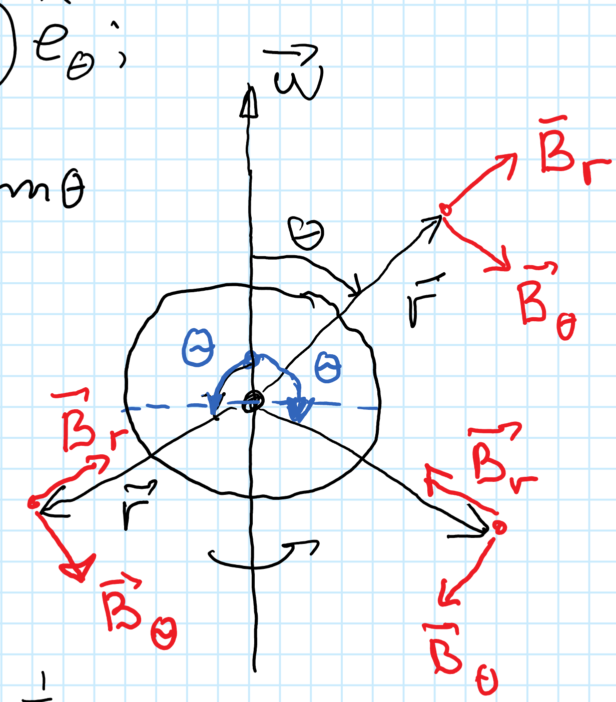
Calculations of the magnetic field $\vec{B}(\vec{r})$ for $r > R$:

$\vec{A} = (\cancel{A_r}, \cancel{A_\theta}, A_\phi)$: $\vec{A} = (0, 0, A_\phi(r,\theta))$

$$\vec{B} = \vec{\nabla}\times\vec{A}(\vec{r}) = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta \cdot A_\phi) - \cancel{\frac{\partial A_\theta}{\partial\phi}}\right]\hat{e}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\cancel{\frac{\partial A_r}{\partial\phi}} - \frac{\partial}{\partial r}(r A_\phi)\right]\hat{e}_\theta$$
$$+ \frac{1}{r}\left[\cancel{\frac{\partial}{\partial r}(r A_\theta)} - \cancel{\frac{\partial}{\partial\theta}A_r}\right]\cdot\hat{e}_\phi$$

$$\vec{B} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \cdot A_\phi(r,\theta))\,\hat{e}_r - \frac{1}{r}\frac{\partial}{\partial r}(r A_\phi(r,\theta))\,\hat{e}_\theta;$$

$$\vec{B} = \hat{e}_r \cdot \frac{2\mu_0 R^4 \omega\sigma}{3 r^3}\cos\theta + \hat{e}_\theta \cdot \frac{\mu_0 R^4 \omega \sigma}{3 r^3}\sin\theta$$



(a) The total energy of EM field inside sphere $r \le R$:

$$W_< = \int d^3r \cdot (\underbrace{w_E}_{0} + \underbrace{w_B}_{\text{const.}}) = \int_0^R 4\pi r^2 dr\, \frac{B^2}{2\mu_0} = \frac{B^2}{2\mu_0}\cdot\frac{4\pi}{3}R^3$$

$$\boxed{W_< = \frac{\left(\frac{2}{3}\mu_0 \sigma R \omega\right)^2}{2\mu_0}\cdot\frac{4\pi}{3}R^3 = \frac{8\mu_0 \sigma^2 R^5 \cdot \pi}{27}\omega^2}$$

$\frac{1}{54\pi}$

(b) Poynting Vector for $r > R$:

$$\boxed{\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0}}$$

$$\vec{E} = k\frac{q}{r^2}\hat{e}_r = \frac{R^2\sigma}{\varepsilon_0 r^2}\hat{e}_r$$

$$\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} = \frac{\frac{R^2\sigma}{\varepsilon_0 r^2}\hat{e}_r \times \frac{\mu_0 R^4 \omega\sigma}{3 r^3}\sin\theta\, \hat{e}_\theta}{\mu_0},$$

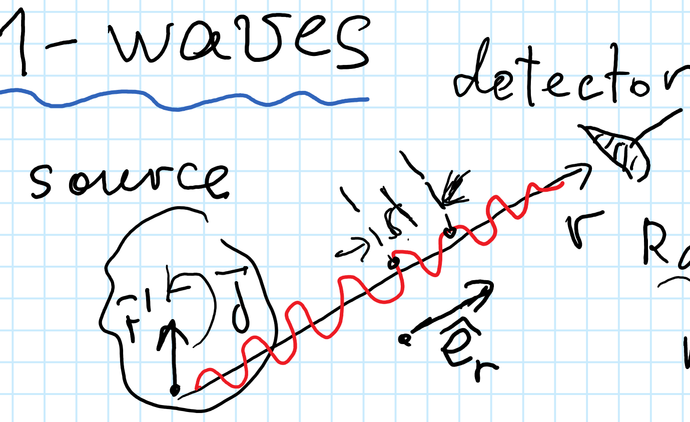
$\hat{e}_r \times \hat{e}_\theta = \hat{e}_\phi$

$\sigma = \frac{q}{4\pi R^2}$

$$\boxed{\vec{S} = \hat{e}_\phi \cdot \frac{\sigma^2 \omega R^6 \cdot \sin\theta}{3\varepsilon_0 r^5} = \hat{e}_\phi \cdot \frac{q^2 \omega \cdot R^2 \sin\theta}{48\pi^2 \varepsilon_0 r^5}}$$

Radiation of EM-waves

Griffith 11.1



Radiation zone:
$r \gg \bar{r}', \lambda$

recap

Solution of wave equation

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\left(\bar{r}', t - \frac{|\bar{r}-\bar{r}'|}{c}\right)}{|\bar{r}-\bar{r}'|}\, d^3r'}$$

↖ vector-potential

Wave length λ

$\lambda = c\cdot T = \frac{2\pi}{k}$

Harmonic sources

Simplest case:

$$\begin{cases} \vec{j}(\vec{r}, t) = \vec{j}_\omega(\vec{r}')\, e^{-i\omega t} & \leftarrow \text{density of electric current} \\ \rho(\vec{r}, t) = \rho_\omega(\vec{r}')\cdot e^{-i\omega t} & \leftarrow \text{density of electric charge} \end{cases}$$

$T = 2\pi/\omega$ - period

$$\boxed{\omega = c\cdot k}$$

Delay time for Radiation Zone

$\tau_d = \frac{|\bar{r}-\bar{r}'|}{c} = \frac{1}{c}\left(r - \hat{e}_r\cdot\vec{r}' + \ldots\right)$; $\hat{e}_r = \frac{\vec{r}}{r}$

$$\boxed{\vec{A}(\vec{r}, t) = \frac{\mu_0\, e^{i(kr-\omega t)}}{4\pi r}\int \vec{j}_\omega(\vec{r}')\cdot e^{-i\vec{k}\vec{r}'}\, d^3r'}$$

Taylor expansion:

small value

$$|\bar{r}-\bar{r}'| = \underset{r \gg r'}{\sqrt{r^2 + r'^2 - 2\bar{r}\bar{r}'}} \simeq r\left(1 - 2\frac{\vec{r}\,\vec{r}'}{r^2} + \cancel{\frac{r'^2}{r^2}}\right)^{1/2} \simeq r - \frac{\vec{r}}{r}\cdot\vec{r}'$$

$$\boxed{\begin{aligned} k &= \frac{\omega}{c} \\ \vec{k} &= k\cdot\hat{e}_r \end{aligned}}$$

$\hat{e}_r \equiv \hat{e}_k$

$k = \frac{2\pi}{\lambda}$

Wave vector $\vec{k}$

We can neglect all small terms in the $\vec{A}$-formula

$kr' = 2\pi\frac{r'}{\lambda} \ll 1$
$(k = 2\pi/\lambda)$

$$\boxed{\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi}\,\frac{e^{i(\vec{k}\vec{r}-\omega t)}}{r}\cdot\int \vec{j}_\omega(\vec{r}')\, d^3r'}$$

This is the expression for $\vec{A}(\bar{r}, t)$ potential in radiation zone