Lecture 12

02/23/2023

recap: We will need this information for analysis of EM-field:

$$\begin{cases} \varphi(\vec{r},t) = k\int \dfrac{\rho(\vec{r}\,', t-\frac{|\vec{r}-\vec{r}\,'|}{c})}{|\vec{r}-\vec{r}\,'|}d^3r' \\ \vec{A}(\vec{r},t) = \dfrac{\mu_0}{4\pi}\int \dfrac{\vec{J}(\vec{r}\,', t-\frac{|\vec{r}-\vec{r}\,'|}{c})}{|\vec{r}-\vec{r}\,'|}d^3r' \end{cases}$$

$$t_r = t - \underbrace{\frac{|\vec{r}-\vec{r}\,'|}{c}}_{\frac{|\vec{r}-\vec{r}\,'|}{c}}$$

$\frac{|\vec{r}-\vec{r}\,'|}{c}$ ← time-delay

Scalar and Vector Potentials of Moving Point Charges (Liénard-Wiechert Potentials)

For the electrostatic field we have introduced the volume density of point charge → $\boxed{\rho(\vec{r}) = q\cdot\delta(\vec{r}-\vec{r}_q)}$

"δ" - is the Dirac delta function

Recapitulation of some properties of the "δ" function:

For 1D space:

$$\int_{-\infty}^{+\infty} f(x')\,\delta(x-x')\,dx' = f(x)$$

$$\int_{-\infty}^{\infty}\delta(x)\,dx = 1$$

For 3D space:

$$\int f(\vec{r}\,')\,\delta(\vec{r}-\vec{r}\,')\,d^3r' = f(\vec{r})$$

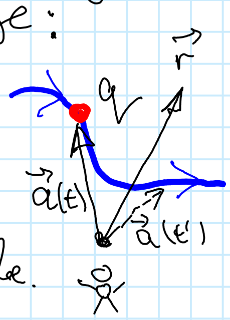
$$\int \delta(\vec{r})\,d^3r = 1$$

$$\delta(\vec{r}) = \delta(x)\cdot\delta(y)\cdot\delta(z)$$

$$\vec{r}(x,y,z)$$

Time-dependent density of the point charge:

$$\boxed{\rho(\vec{r},t) = q\,\delta(\vec{r}-\vec{a}(t))}$$



$\vec{a}(t)$ is the radius-vector of our charge particle.

"t" is the real time.

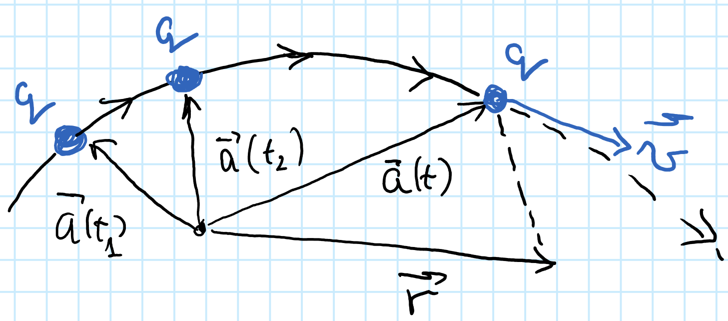
The radius vector $\vec{a}(t)$ is considered as a known function of the time t. The scalar potential induced by moving point charge "q" can be written as:

$$t_r = t - \frac{|\vec{r}-\vec{r}\,'|}{c}$$

$$\varphi(\vec{r},t) = k\int\frac{\rho(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|}d^3r' = k\int\frac{q\,\delta(\vec{r}\,'-\vec{a}(t_r))}{|\vec{r}-\vec{r}\,'|}d^3r' = kq\int\frac{\delta(\vec{r}\,'-\vec{a}(t-\frac{|\vec{r}-\vec{r}\,'|}{c}))}{|\vec{r}-\vec{r}\,'|}d^3r'$$

$$\begin{cases} t_r = t - \dfrac{|\vec{r}-\vec{r}\,'|}{c} \\ \vec{a}(t_r) \equiv \vec{r}\,'(t_r) \end{cases}$$

time delay



The radius vector $\vec{a}(t)$ is considered as a known function of the time t. The scalar potential $\varphi(\vec{r},t)$ induced by moving point charge "q"

Math. recapitulation:

Expression for the potential can be simplified. We will need to do a few transformations, which involve the δ-function and it's property.

Integration with the δ-function is usually a very simple procedure:

$$\int f(\vec{r}\,')\,\delta(\vec{r}\,'-\vec{r}_0)\,d^3\vec{r}\,' = f(\vec{r}_0) \quad [\vec{r}\,'\to\vec{r}_0]$$

In our case: $\vec{r}\,' \equiv \vec{a}\left(t-\frac{|\vec{r}-\vec{r}\,'|}{c}\right)$

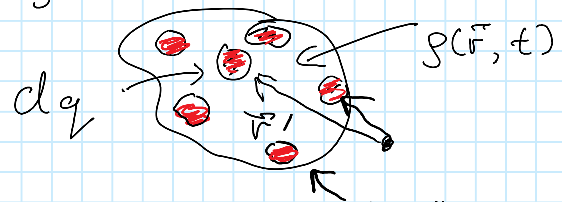
and we have to solve this equation to determine $\vec{r}_0\,' \equiv \vec{r}_0\,'(t)$!!

$\boxed{\text{Step 1}}$ Let us to consider an expression for the volume charge density, $\rho(\vec{r},t)$. We have defined the volume charge density as a time-dependent function $\rho(\vec{r},t)$, wich describes a local charge distribution:

$dq = \rho(\vec{r},t)\cdot d^3r$

and

$$\boxed{q = \int dq = \int \rho(\vec{r}\,';t)\,d^3r'}$$



The time "t" has the same value for all small "dq" charges.

In our expression for the potential: the $\rho(\vec{r}\,',t_r)$ density depends on the delayed time $t_r = t-\frac{|\vec{r}-\vec{r}\,'|}{c}$; $\rho(\vec{r}\,';t_r) = \rho\left(\vec{r}\,';t-\frac{|\vec{r}-\vec{r}\,'|}{c}\right)$.

$$\boxed{\int_V \rho(\vec{r}\,';t_r)\,d^3r' \neq q}$$

The static limit "$c\to\infty$"

$\rho\left(\vec{r}\,';t-\frac{|\vec{r}-\vec{r}\,'|}{c}\right) \simeq \rho(\vec{r}\,';t)$ if $c\to\infty$

Calculations of the volume integral cannot yield the total charge q because the density $\rho(\vec{r}\,';t_r)$ includes an additional dependance on $\vec{r}\,'$ $\boxed{t_r = t-\frac{|\vec{r}-\vec{r}\,'|}{c}}$. The time delay $\boxed{\tau_d = \frac{|\vec{r}-\vec{r}\,'|}{c}}$ is different for different points

$\boxed{\text{Step 2}}$

Calculations of the potential of a moving point charge

$$\varphi(\vec{r},t) = K\int \frac{\rho(\vec{r}\,';t_r)}{|\vec{r}-\vec{r}\,'|}\,d^3r' = K\iint \underbrace{\frac{\rho(\vec{r}\,';t')}{|\vec{r}-\vec{r}\,'|}\,d^3r'\,dt'\,\delta(t'-t_r)}_{\text{expression with two delta-functions}}$$

where $t_r = t-\frac{|\vec{r}-\vec{r}\,'|}{c}$

This equation is valid for any charge density, and we will use it for the density of point charges.

$$\varphi(\vec r,t)=kq\iint \frac{d^3r'\,dt'\,\delta(\vec r\,'-\vec a(t'))}{|\vec r-\vec r\,'|}\,\delta\Big(\underbrace{t'-t+\frac{|\vec r-\vec r\,'|}{c}}_{t'-t_r}\Big)$$

In this equation: $-\infty\le t'\le\infty$ and $a(t')$ does not depend on t_r !!

Integration over volume coordinates $\vec r\,'$:

$$\varphi(\vec r,t)=kq\int dt'\int \frac{d^3r'}{|\vec r-\vec r\,'|}\,\delta(\vec r\,'-\vec a(t'))\,\delta\left(t'-t+\frac{|\vec r-\vec r\,'|}{c}\right)$$

Integration of the δ-function: $\boxed{\int f(\vec r\,')\,\delta(\vec r\,'-\vec r_0)\,d^3r'=f(\vec r_0)}$ replacement $\vec r\,'\to\vec r_0$

In our integration it should be $\boxed{\vec r\,'\to\vec a(t')}$

$\Rightarrow$

$$\varphi(\vec r,t)=kq\int_{-\infty}^{+\infty}dt'\,\frac{\delta\left(t'-t+\frac{|\vec r-\vec a(t')|}{c}\right)}{|\vec r-\vec a(t')|}$$

Integration: $\displaystyle\int_{-\infty}^{+\infty}dt'\,\delta\left(t'-t+\frac{|\vec r-\vec a(t')|}{c}\right)$

Math $\to$ $\displaystyle\int_{-\infty}^{+\infty}\delta[f(x)]\,dx=\frac{1}{|f'(x_0)|}$, where $f(x=x_0)=0$

$\boxed{\displaystyle\int_{-\infty}^{+\infty}\delta[f(x)]\,dx=\sum_i\frac{1}{|f'(x_i)|}\ \text{for several roots } x_i}$

Proof: $\displaystyle\int\delta(ax)\,dx=\frac{1}{|a|}$;

$$f(x)=\cancelto{0}{f(x_0)}+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\ldots$$

The Taylor expantion near the root point x_0.

$$\int\delta[f(x)]\,dx=\int\delta[\cancel{f(x_0)}+f'(x_0)(x-x_0)]\,dx=$$

$$=\int\delta[f'(x_0)\cdot(x-\overset{0}{x_0})]\,dx=\frac{1}{|f'(x_0)|};$$

Our Integral can be calculated:

$$\int_{-\infty}^{+\infty}\frac{dt'\cdot\delta\left[t'-t+\frac{|\vec r-\vec a(t')|}{c}\right]}{|\vec r-a(t')|}=\frac{1}{|\dot f(t_0')|\cdot|\vec r-\vec a(t_0')|}$$

root where $\dot f=\dfrac{df}{dt'}$

Math. $\to$ $\displaystyle\int_{-\infty}^{+\infty}\frac{dt'\cdot\delta[f(t')]}{|\vec r-\vec a(t')|}$, where $\boxed{f(t')=t'-t+\frac{|\vec r-\vec a(t')|}{c}}$

Root: $f(t_0')=0\Rightarrow t_0'=\underbrace{t-\frac{|\vec r-\vec a(t_0')|}{c}}=t_r\ \Rightarrow\ t_0'=t_r$ Root

This is the time t_r which includes a time delay: $t_r=t-\dfrac{|\vec r-\vec a(t_r)|}{c}$

$$\dot f(t')=\frac{df}{dt'}=1+\frac{\partial}{\partial t'}\left(\frac{|\vec r-\vec a(t')|}{c}\right)$$

$$\dot f(t_r)=1+\frac{1}{c}\left(\frac{\partial}{\partial t'}|\vec r-\vec a(t')|\right)_{t'=t_r}=1-\frac{1}{c}\frac{\partial\vec a(t')}{\partial t'}\bigg|_{t'=t_r}\cdot\vec\nabla_{\vec r}|\vec r-\vec a(t_r)|=$$

$$=1-\frac{\vec v(t_r)}{c}\cdot\frac{\vec r-\vec a(t_r)}{|\vec r-\vec a(t_r)|}=1-\frac{\vec n_r\vec v(t_r)}{c},\quad \vec n_r=\frac{\vec r-\vec a(t_r)}{|\vec r-\vec a(t_r)|}$$

unit vector

The final formula for the scalar potential of moving charge is given by

$$\boxed{\varphi(\vec{r},t) = k\,\frac{q}{|\vec{r}-\vec{a}(t_r)|\cdot\left(1-\dfrac{\vec{n}\vec{v}(t_r)}{c}\right)}}$$

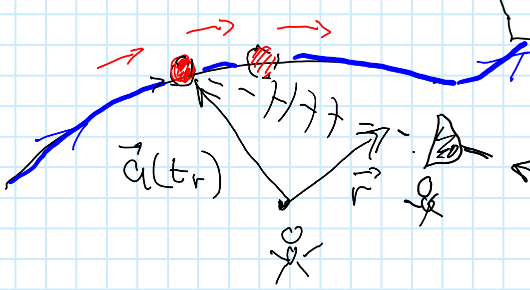
Liénart-Wiechert scalar potential

Delayed time: $t_r = t - \dfrac{|\vec{r}-\vec{a}(t_r)|}{c}$

$$\boxed{t_r + \frac{|\vec{r}-\vec{a}(t_r)|}{c} = t}$$

This is the equation for the calculation of t_r-value

$$\boxed{\vec{n} = \frac{\vec{r}-\vec{a}(t_r)}{|\vec{r}-\vec{a}(t_r)|}}$$



unit vector of the direction between detector and particle at the time t_r.

The vector-potential of moving point charge: $\vec{j}(\vec{r},t) = \vec{v}\rho(\vec{r},t) = \vec{v}q\,\delta(\vec{r}-\vec{a}(t))$

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\,\frac{q\vec{v}(t_r)}{|\vec{r}-\vec{a}(t_r)|\left(1-\dfrac{\vec{n}\vec{v}(t_r)}{c}\right)}}$$

Lienart-Wiechert vector-potential

$$\boxed{\vec{v} = \frac{d\vec{a}(t)}{dt}} \qquad \boxed{\vec{A}(\vec{r},t) = \frac{\vec{v}(t_r)}{c^2}\,\varphi(\vec{r},t_r)}$$

"t_r" can be calculated from the equation

$$\boxed{t_r + \frac{|\vec{r}-\vec{a}(t_r)|}{c} = t}$$

If $\vec{r} = 0$

$$\boxed{t_r + \frac{a(t_r)}{c} = t}$$