Lecture 11

02/21/2023

Independent reading of Chapters:
Griffiths textbook → 9.2; 9.3.1–2

Retarded Potentials (Griffiths, 10.2.1)

Recapitulation: wave equations for the scalar and vector potentials

$$\boxed{\vec{\nabla}^2 \varphi(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \varphi(\vec{r},t)}{\partial t^2} = -\frac{\rho(\vec{r},t)}{\varepsilon_0}}$$

and

$$\boxed{\vec{\nabla}^2 \vec{A}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \vec{A}(\vec{r},t)}{\partial t^2} = -\mu_0 \vec{j}(\vec{r},t)}$$

These equations have been obtained from Maxwell's equation and the Lorenz gauge conditions.

$$\boxed{\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0}$$

Solution of the wave equations for φ and $\vec{A}$ potentials.

Lorenz gauge $\vec{\nabla}\vec{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0$

$$\begin{cases} \vec{\nabla}^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\rho/\varepsilon_0 \\ \vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{j} \end{cases}$$

in the vacuum $\rho = 0$ and $\vec{j} = 0$

$$\begin{cases} \vec{\nabla}^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 0 \\ \vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = 0 \end{cases}$$

solutions: harmonic waves

$$\varphi(\vec{r},t) = e^{-i\omega t}\varphi_\omega(\vec{r})$$
$$\vec{A}(\vec{r},t) = e^{-i\omega t}\cdot\vec{A}_\omega(\vec{r})$$

$$\Rightarrow \begin{cases} \vec{\nabla}^2\varphi_\omega(\vec{r}) + \left(\frac{\omega}{c}\right)^2\varphi_\omega(\vec{r}) = 0 \;\}\; \varphi_\omega(\vec{r}) = \varphi_\omega\cdot e^{i\vec{k}\vec{r}} \\ \vec{\nabla}^2\vec{A}_\omega(\vec{r}) + \left(\frac{\omega}{c}\right)^2\vec{A}_\omega(\vec{r}) = 0 \;\}\; \vec{A}_\omega(\vec{r}) = A_\omega e^{i\vec{k}\vec{r}} \end{cases}$$

the wave vector $\boxed{\vec{k} = \omega/c\cdot\hat{e}_k}$

Solutions:

$$\boxed{\begin{aligned} \varphi(\vec{r},t) &= \varphi_\omega\cdot e^{i(\vec{k}\vec{r}-\omega t)} \\ \vec{A}(\vec{r},t) &= \vec{A}_\omega\cdot e^{i(\vec{k}\vec{r}-\omega t)} \end{aligned}}$$

harmonic wave

Electric field: $\vec{E} = -\frac{\partial A}{\partial t} - \vec{\nabla}\varphi = +i\omega e^{i(\vec{k}\vec{r}-\omega t)}\vec{A}_\omega - i\vec{k}\, e^{i(\vec{k}\vec{r}-\omega t)}\cdot\varphi_\omega$

$\boxed{\vec{B} = i\vec{k}\times\vec{A}(\vec{r},t)}$

$$\vec{B} = \vec{\nabla}\times\vec{A}(\vec{r},t) = e^{-i\omega t}\cdot\vec{\nabla}_r\times(\underset{a}{e^{i\vec{k}\vec{r}}}\cdot\underset{b}{\vec{A}_\omega}) = i\vec{k}\times\underbrace{e^{i(\vec{k}\vec{r}-\omega t)}\cdot\vec{A}_\omega}_{\vec{A}(\vec{r},t)}$$

Math.: $\vec{\nabla}_r\times(a(\vec{r})\cdot\vec{b}(\vec{r})) = (\vec{\nabla}\cdot a(\vec{r}))\times\vec{b}(\vec{r}) + a(\vec{r})\,\vec{\nabla}\times\vec{b}(\vec{r})$

$\boxed{\vec{E} = c(\vec{B}\times\hat{e}_k)}$

$\omega = ck$

$\vec{E}$ and $\vec{B}$ should be real function

$$\vec{E} = \mathrm{Re}\{-i\omega\vec{A}\} = \omega\vec{A}_\omega\cdot\sin(\vec{k}\vec{r}-\omega t)$$
$$\vec{B} = \mathrm{Re}\{i\vec{k}\times\vec{A}\} = -\vec{k}\times\vec{A}_\omega\sin(\vec{k}\vec{r}-\omega t)$$

Retarded Potentials (Griffiths, 10.2.1)

Recapitulation: wave equations for the scalar and vector potentials

$$\boxed{\vec{\nabla}^2 \varphi(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \varphi(\vec{r},t)}{\partial t^2} = -\frac{\rho(\vec{r},t)}{\varepsilon_0}}$$

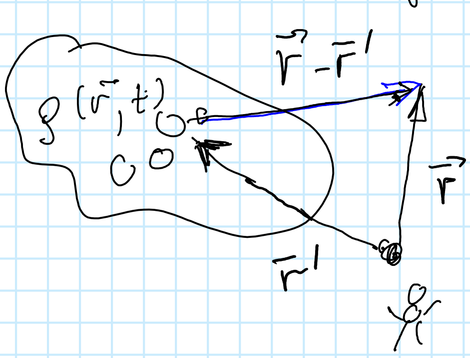
and

$$\boxed{\vec{\nabla}^2 \vec{A}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2 \vec{A}(\vec{r},t)}{\partial t^2} = -\mu_0 \vec{j}(\vec{r},t)}$$

These equations have been obtained from Maxwell's equation and the Lorenz gauge conditions.

$$\boxed{\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0}$$

The solutions of these non-homogeneous equations can be obtained using "retarded potentials".
Physical meaning of retarded potentials: they take into account the "retarded time" t_r



$\boxed{t_r = t - \frac{|\vec{r}-\vec{r}\,'|}{c}}$ where $\frac{|\vec{r}-\vec{r}\,'|}{c}$ is the time required for the EM field to travel from the source point $\vec{r}\,'$ to the location of observational point $\vec{r}$

Solution of the time-independent Poisson's equation (recapitulation)

electrostatic potential $\varphi_s(\vec{r})$ and Poisson's equation

Static equation: $\boxed{\vec{\nabla}_r^2 \varphi_s(\vec{r}) = -\rho(\vec{r})/\varepsilon_0} \Rightarrow \boxed{\varphi = k\int \frac{\rho(\vec{r}\,')\,d^3r'}{|\vec{r}-\vec{r}\,'|}}$ $\quad k = \frac{1}{4\pi\varepsilon_0}$

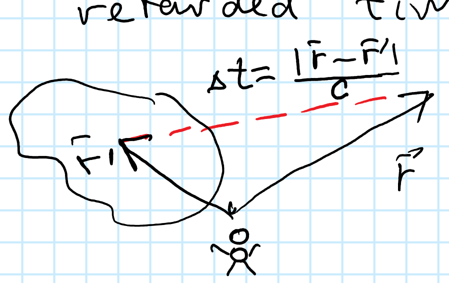
Solution of the time-dependent wave equation for the scalar potential $\varphi(\vec{r},t)$ can be written as:

$$\boxed{\varphi(\vec{r},t) = k\int \frac{\rho\left(\vec{r}\,', t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{|\vec{r}-\vec{r}\,'|}\, d^3r'}$$

retarded potential

$$\boxed{t_r = t - \frac{|\vec{r}-\vec{r}\,'|}{c}}$$

retarded time t_r



(Retardation was obtained in our solutions $f_+(x-ct) \equiv g_+\left(t-\frac{x}{c}\right)$ of the one-dimensional wave equation.)

We have to show, that the retarded potential $\varphi(\vec r, t)$ satisfies to the wave equation:

$$\boxed{\vec\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t} = -\frac{\rho(\vec r, t)}{\varepsilon_0}}$$

Let's calculate $\vec\nabla^2\varphi$:

$$K = 1/4\pi\varepsilon_0$$

$$\vec\nabla_r\cdot\varphi\left(\vec r, t - \frac{|\vec r - \vec r\,'|}{c}\right) = \vec\nabla_r\left(K\int\frac{\rho(\vec r\,', \overbrace{t - \frac{|\vec r - \vec r\,'|}{c}}^{t_r})}{|\vec r - \vec r\,'|}d^3r'\right) =$$

$$= K\int\vec\nabla_r\left(\frac{1}{|\vec r - \vec r\,'|}\right)\rho\left(\vec r\,', t - \frac{|\vec r - \vec r\,'|}{c}\right)d^3r' + K\int\frac{d^3r'}{|\vec r - \vec r\,'|}\vec\nabla_r\rho\left(\vec r, t - \frac{|\vec r - \vec r\,'|}{c}\right);$$

$$\vec\nabla^2\varphi\left(\vec r, t - \frac{|\vec r - \vec r\,'|}{c}\right) = K\int\vec\nabla^2\left(\frac{1}{|\vec r - \vec r\,'|}\right)\rho\left(\vec r\,', t - \frac{|\vec r - \vec r\,'|}{c}\right)d^3r' + K\int\frac{d^3r'}{|\vec r - \vec r\,'|}\vec\nabla_r^2\rho\left(\vec r\,', t - \frac{|\vec r - \vec r\,'|}{c}\right) +$$

$$+ 2K\int\vec\nabla_r\left(\frac{1}{|\vec r - \vec r\,'|}\right)\cdot\vec\nabla_r\varphi\left(\vec r\,', t - \frac{|\vec r - \vec r\,'|}{c}\right)d^3r;$$

This term gives the δ-function $\vec\nabla^2\left(\frac{1}{|\vec r - \vec r\,'|}\right) = -4\pi\delta(\vec r - \vec r\,')$

Notations: $\dot\rho = \frac{\partial\rho}{\partial t}$

$$\vec\nabla\left(|\vec r - \vec r\,'|\right) = \frac{\vec r - \vec r\,'}{|\vec r - \vec r\,'|} = \hat e_{rr'}$$

$$\vec\nabla\cdot\hat e_{rr'} = \frac{2}{|\vec r - \vec r\,'|}$$

$$\vec\nabla^2\varphi\left(\vec r, t - \frac{|\vec r - \vec r\,'|}{c}\right) = -\frac{\rho(\vec r, t)}{\varepsilon_0} +$$

$$+ 2K\int\vec\nabla_r\left(\frac{1}{|\vec r - \vec r\,'|}\right)\cdot\vec\nabla_r\left(-\frac{|\vec r - \vec r\,'|}{c}\right)\dot\rho(\vec r\,', t_r)\,d^3r' +$$

$$-\frac{1}{c}\frac{\vec r - \vec r\,'}{|\vec r - \vec r\,'|}$$

$$+ K\int\frac{d^3r'}{|\vec r - \vec r\,'|}\vec\nabla_r\left(\dot\rho(\vec r\,', \underbrace{t - \frac{|\vec r - \vec r\,'|}{c}}_{t_r})\cdot\frac{\vec r - \vec r\,'}{|\vec r - \vec r\,'|}\frac{(-1)}{c}\right) =$$

$$= -\frac{\rho(\vec r, t)}{\varepsilon_0} + 2\frac{K}{c}\int d^3r'\frac{(\vec r - \vec r\,')^2}{|\vec r - \vec r\,'|^4}\dot\rho(\vec r\,', t_r) - \frac{K}{c}\int\frac{d^3r'}{|\vec r - \vec r\,'|}\dot\rho(t, t_r)\,\vec\nabla_r\left(\frac{\vec r - \vec r\,'}{|\vec r - \vec r\,'|}\right) +$$

$$+ \frac{K}{c^2}\int\frac{d^3r'}{|\vec r - \vec r\,'|}\frac{\cancel{(\vec r - \vec r\,')^2}}{\cancel{|\vec r - \vec r\,'|^2}}\ddot\rho\left(\vec r\,', t - \frac{|\vec r - \vec r\,'|}{c}\right);$$

$$\vec\nabla_r\left(\frac{\vec r - \vec r\,'}{|\vec r - \vec r\,'|}\right) = \frac{3}{|\vec r - \vec r\,'|} - \frac{(\vec r - \vec r\,')^2}{|\vec r - \vec r\,'|^3} = \frac{2}{|\vec r - \vec r\,'|}$$

$$\Rightarrow \vec\nabla^2\varphi(\vec r, t) = -\frac{\rho(\vec r, t)}{\varepsilon_0} + \frac{2K}{c}\int d^3r'\cancelto{0}{\left[\frac{\dot\rho(\vec r\,', t_r)}{|\vec r - \vec r\,'|^2} - \frac{\dot\rho(\vec r\,', t_r)}{|\vec r - \vec r\,'|^2}\right]} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\underbrace{\left(K\int\frac{\rho\left(\vec r\,', t - \frac{|\vec r - \vec r\,'|}{c}\right)}{|\vec r - \vec r\,'|}d^3r'\right)}_{\varphi(\vec r, t)}$$

Finally:

$$\boxed{\vec\nabla^2\varphi(\vec r, t) - \frac{1}{c^2}\frac{\partial^2\varphi(\vec r, t)}{\partial t} = -\frac{\rho(\vec r, t)}{\varepsilon_0}}$$

where $\boxed{\varphi = K\int\frac{\rho\left(\vec r\,', t - \frac{|\vec r - \vec r\,'|}{c}\right)}{|\vec r - \vec r\,'|}d^3r'}$ is the retarded scalar potential.

We can show, using results of our calculations for the scalar potential, that:

$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\left(\vec{r}\,', t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{|\vec{r}-\vec{r}\,'|}\, d^3r'} \quad (*)$$

The retarded vector potential (*) is an exact solution of the wave equation

$$\boxed{\vec{\nabla}^2 \cdot \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{j}(\vec{r},t)} \quad (**)$$

Harmonic waves of potentials.

$$\begin{cases} \vec{J} = J_\omega(\vec{r}\,)\, e^{-i\omega t} \\ \rho(\vec{r},t) = \rho_\omega(\vec{r}\,)\, e^{-i\omega t} \end{cases}$$

Solution of the wave equation for harmonic sources $\vec{J}(\vec{r},t)$ and $\rho(\vec{r},t)$

Retarded potential $\vec{A}(\vec{r},t)$:

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\left(\vec{r}\,', t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{|\vec{r}-\vec{r}\,'|}\, d^3r' \Rightarrow$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}_\omega(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, e^{-i\omega\left(t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)} d^3r' =$$

$$= \frac{\mu_0}{4\pi} e^{-i\omega t}\int \frac{\vec{J}_\omega(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, e^{+i\frac{\omega}{c}|\vec{r}-\vec{r}\,'|} d^3r'$$

$$\boxed{\vec{A}(\vec{r},t) = \vec{A}_\omega(\vec{r}\,)\cdot e^{-i\omega t}} \quad \text{where} \quad \boxed{\vec{A}_\omega(\vec{r}\,) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')\, e^{-ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\, d^3r'}$$