Lecture 10

02/16/2023

recap:

wave equations
$$\begin{cases} \vec{\nabla}^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0 \\ \vec{\nabla}^2\vec{B} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = 0 \end{cases}$$

Potentials: $\varphi, \vec{A}$

$$\begin{cases} \vec{E}(\vec{r},t) = -\frac{\partial\vec{A}(\vec{r},t)}{\partial t} - \vec{\nabla}\varphi(\vec{r},t) \\ \vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A}(\vec{r},t) \end{cases}$$

Wave Equations for the Scalar $\varphi(\vec{r},t)$ and Vector $\vec{A}(\vec{r},t)$ Potentials

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Maxwell's equations: $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ and $\vec{\nabla}\cdot\vec{B} = 0$

source equations

Dynamic Equations:
$$\begin{cases} \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow \text{from this equation} \\ \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \end{cases}$$

Formulas for the vector-potential $\boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$ and $\boxed{\vec{E} = -\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\varphi}$

From the source equation:

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\left(-\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\varphi\right) = -\frac{\partial}{\partial t}\vec{\nabla}\vec{A} - \vec{\nabla}^2\varphi = \rho/\varepsilon_0$$

$$\Rightarrow \boxed{\vec{\nabla}^2\varphi + \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = -\rho/\varepsilon_0} \quad \text{Eq. 1}$$

$$\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = \underset{a}{\vec{\nabla}}\times(\underset{b}{\vec{\nabla}}\times\underset{c}{\vec{A}}) = \mu_0\vec{J} + \mu_0\varepsilon_0\left(-\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\frac{\partial\varphi}{\partial t}\right)$$

$$\vec{\nabla}(\vec{\nabla}\vec{A}) - \vec{\nabla}^2\vec{A} = \mu_0\vec{J} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \frac{1}{c^2}\vec{\nabla}\frac{\partial\varphi}{\partial t}$$

The wave equation for $\vec{A}(\vec{r},t)$:

$$\boxed{\vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\cdot\left(\vec{\nabla}\vec{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t}\right) = -\mu_0\vec{J}} \quad \text{Eq. 2}$$

Equations (1) and (2) describe dynamic changes of the electric and magnetic fields and their interaction with charged particles.

The equations (*) and (**) include terms mixing $\vec{A}$ and φ potentials.

$$\boxed{\begin{aligned} &\vec{\nabla}^2\varphi + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{\rho(\vec{r},t)}{\varepsilon_0} \\ &\vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} + \vec{\nabla}\cdot\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t}\right) \end{aligned}}$$

ρ

$\vec{J} = \rho\cdot\vec{v}$

Gauge Transformation

$\vec{B} = \vec{\nabla}\times\vec{A}$. Could we change $\boxed{\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\chi(\vec{r},t)}$?

$$\vec{B}' = \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} + \cancel{\vec{\nabla}\times\vec{\nabla}\chi}^{\,0} \Rightarrow \vec{B}' = \vec{B}$$

The vector potential is not uniquely defined: $\vec{A}$ and $\vec{A}'$ yield the same magnetic field

The electric field depends both on φ and $\vec{A}$: $\boxed{\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\varphi}$

$\vec{E}$-field for the $\vec{A}'$-potential: $\vec{E}' = -\frac{\partial \vec{A}'}{\partial t} - \vec{\nabla}\varphi = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\varphi - \vec{\nabla}\frac{\partial \chi}{\partial t}$

$$\vec{E}' = \underbrace{-\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\varphi}_{\vec{E}} - \vec{\nabla}\frac{\partial\chi}{\partial t} = \vec{E} - \vec{\nabla}\frac{\partial \chi}{\partial t} \Rightarrow \vec{E} \neq \vec{E}'$$

New electric field $\vec{E}'$ is not equal to the old electric field $\vec{E}$.

For the $\vec{A} \to \vec{A}'$ transformation:

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\varphi$$

$\vec{B}$-field is an invariant of $\vec{A} \to \vec{A}'$ transformation

$\vec{E}$-field? How should we transform φ-potential, if $\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\chi(\vec{r},t)$

$\vec{E}' = \vec{E}$

$\vec{B}' = \vec{B}$

We can find a consistant transformations of $\vec{A}$ an φ potentials, which keep $\vec{E}$ and $\vec{B}$ vectors unchanged.

$$\begin{cases} \vec{A}' = \vec{A} + \vec{\nabla}\chi(\vec{r},t) \\ \varphi' = \varphi - \frac{\partial \chi}{\partial t} \end{cases}$$

$$\vec{E}' = -\frac{\partial \vec{A}'}{\partial t} - \vec{\nabla}\varphi' = -\frac{\partial \vec{A}}{\partial t} - \cancel{\vec{\nabla}\frac{\partial \chi}{\partial t}} - \vec{\nabla}\varphi + \cancel{\vec{\nabla}\frac{\partial \chi}{\partial t}} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\varphi \equiv \vec{E}$$

Gauge invariance → $\boxed{\vec{E}' = \vec{E}}$ and $\boxed{\vec{B}' = \vec{B}}$

if

Gauge Transformation → $\boxed{\begin{array}{l} \vec{A}' = \vec{A} + \vec{\nabla}\chi(\vec{r},t) \\ \varphi' = \varphi - \frac{\partial \chi(\vec{r},t)}{\partial t} \end{array}}$

The gauge trasformation allows to change the expressions for the φ and $\vec{A}$ potentials without changes of $\vec{E}$ and $\vec{B}$ vectors.

Gauge function $\chi(\vec{r},t)$ can be selected to simplify Eq. 1 and Eq. 2. Two types of χ-function are especially useful for analysis of the ED wave equations:

(a) The Lorenz Conditions.

$$\boxed{\vec{\nabla}\vec{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0} \Rightarrow \boxed{\vec{\nabla}\vec{A} = -\frac{1}{c^2}\frac{\partial \varphi}{\partial t}}$$

Substituting this formula in Eq. 1

We can derive the wave equation for the scalar potential φ:

$$\vec{\nabla}^2\varphi + \frac{\partial}{\partial t}\vec{\nabla}\vec{A} = -\rho/\varepsilon_0 \Rightarrow \boxed{\vec{\nabla}^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\rho/\varepsilon_0} \quad \text{Eq. 3}$$

Lorenz and Lorentz

and for the vector-potential $\vec{A}$:

$$\boxed{\vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{j}} \quad \text{Eq. 4}$$

If $\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0$, the gauge transformation yields $\vec{A}'$ and φ' which could satisfy to the Lorenz condition, if $\boxed{\vec{\nabla}^2\chi - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2} = 0}$. This can be shown directly from definition of the Lorenz conditions:

$$\vec{\nabla}\vec{A}' + \frac{1}{c^2}\frac{\partial\varphi'}{\partial t} = \vec{\nabla}\vec{A} + \vec{\nabla}^2\chi + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2} = \underbrace{\left(\vec{\nabla}\vec{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t}\right)}_{0} + \underbrace{\vec{\nabla}^2\chi - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2}}_{0} = 0$$

$\vec{A}'$ and φ' satisfy to the Lorenz conditions, if $\vec{\nabla}^2\chi - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2} = 0$ (wave equation for the gauge function χ).

Gauge transformation allows to obtaine $\vec{A}'$ and φ' satisfying to the Lorenz conditions, if $\vec{A}$ and φ do not satisfy them:

$$\vec{\nabla}\vec{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = \underset{\text{some given function}}{F(\vec{r},t)} \neq 0$$

$$\vec{\nabla}\vec{A}' + \frac{1}{c^2}\frac{\partial\varphi'}{\partial t} = \underbrace{\left(\vec{\nabla}\vec{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t}\right)}_{F(\vec{r},t)} + \vec{\nabla}^2\chi - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2} = F(\vec{r},t) + \vec{\nabla}^2\chi - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2}$$

We can select the gauge function $\chi(\vec{r},t)$ which is a solution the equation:

$$\vec{\nabla}^2\chi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -F(\vec{r},t).$$

In this case $\vec{\nabla}\vec{A}' + \frac{1}{c^2}\frac{\partial\varphi'}{\partial t} = 0$

(b) The Coulomb conditions

$\boxed{\vec{\nabla}\cdot\vec{A}=0}$ The Eq.(1) is simplified as:

$$\vec{\nabla}^2\varphi+\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A})=-\rho/\varepsilon_0 \Rightarrow \boxed{\vec{\nabla}^2\varphi=-\frac{\rho(\vec{r},t)}{\varepsilon_0}} \quad \text{Eq. 5}$$

The Eq.5 is Poisson's equation with the time-dependent charge density $\rho(\vec{r},t)$. The solution of Eq.5 $\rightarrow$ $\boxed{\varphi(\vec{r},t)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}d^3r'}$ can be substituted into Eq. 2 for the vector potential $\vec{A}$

$$\vec{\nabla}^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\mu_0\vec{j}+\frac{1}{c^2}\vec{\nabla}_r\frac{\partial\varphi}{\partial t};$$

$$\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|}=-\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}=-\frac{\hat{e}_{rr'}}{|\vec{r}-\vec{r}'|^2}$$

$$\boxed{\vec{\nabla}^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\mu_0\vec{j}(\vec{r},t)-\frac{\mu_0}{4\pi}\int\frac{\frac{\partial\rho(\vec{r}',t)}{\partial t}\cdot\hat{e}_{rr'}}{|\vec{r}-\vec{r}'|^2}d^3r'} \quad \text{Eq. 6}$$