02/14/2023

Lecture 9

$$\boxed{w = \frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}}$$

recap:

Poynting vector $\vec{S}$

Eq.(*) $$\boxed{\frac{\partial}{\partial t} w + \vec{\nabla}\left(\frac{\vec{E}\times\vec{B}}{\mu_0}\right) = -\vec{j}\cdot\vec{E}}$$

Joule's heating density (power density):

$$\boxed{\frac{dP}{d^3r} = \vec{j}\cdot\vec{E}}$$

The Joule's heating transfers the energy of EM field to the kinetic energy of charged particles.

Integral form of Eq.(*):

$$\frac{\partial}{\partial t}\Big(\underbrace{\int_V w\, d^3r}_{}\Big) + \int_V \underbrace{d^3r \cdot \vec{\nabla}\cdot\vec{S}}_{} = -\int_V \vec{j}\vec{E}\, d^3r$$

Total energy of the EM field

Gauss's theorem:

The flux of $\vec{E}\times\vec{B}/\mu_0$ vector

$$\boxed{\int_V d^3r \cdot \vec{\nabla}\left(\frac{\vec{E}\times\vec{B}}{\mu_0}\right) = \oint_S \frac{\vec{E}\times\vec{B}}{\mu_0}\cdot d\vec{a}}$$

$$\boxed{\frac{\partial W}{\partial t} = -\oint_S \vec{S}\, d\vec{a} - \int_V \vec{j}\cdot\vec{E}\, d^3r}$$

$$W = \int_V w\, d^3r = \int\left(\frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}\right) d^3r$$

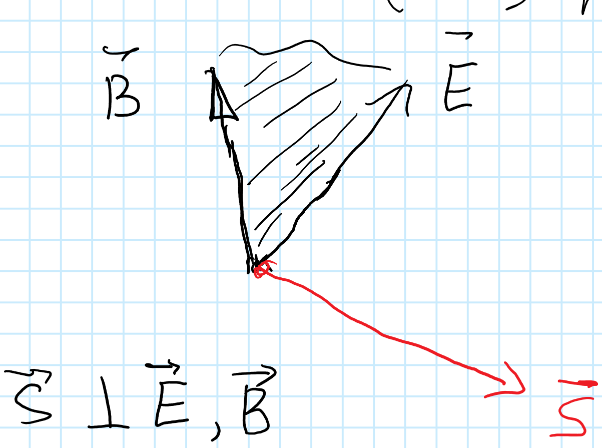
the energy of EM field in the volume V

$$\boxed{\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0}}$$ ← the vector of the EM energy flux

$$\boxed{\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0}}$$ ← Poynting vector

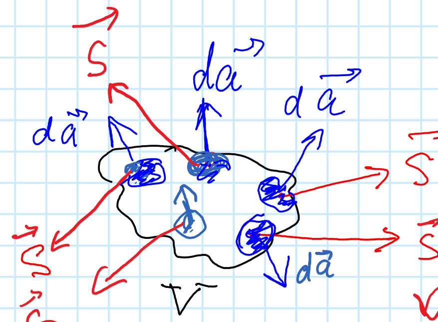
$$\boxed{\vec{S} = c\,(w_E + w_B)\,\hat{e}_s}$$

Physical meaning: the EM energy, crossing unit area (m^2) per second



$\vec{S} \perp \vec{E}, \vec{B}$

Direction of the $\vec{S}$-vector is the direction of propagation of the energy of EM field



Conservation of the EM energy:

Changes of the total energy W:

$$\frac{\partial w}{\partial t} + \vec{\nabla}\vec{S} = -\vec{j}\vec{E}$$

integral form

$$\boxed{W = \int_V d^3r\, w}$$

differential form

$$w = \frac{\varepsilon_0 E^2}{2} + B^2/2\mu_0$$

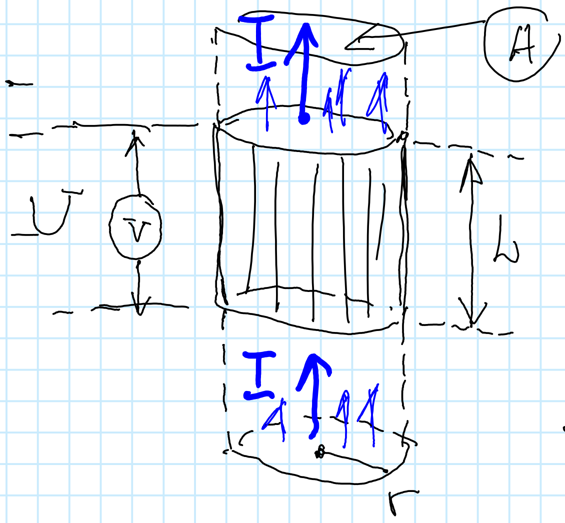
$$\vec{S} = \vec{E}\times\vec{B}/\mu_0$$

$$\boxed{\frac{\partial W}{\partial t} = -\underbrace{\int_{(S)} \vec{S}\, d\vec{a}}_{} - \int \vec{j}\cdot\vec{E}\, d^3r}$$

the energy flux through the boundary "S"

the EM energy converted per second to the kinetic energy of charged particles, i.e. into Joule's heat.

Problem: Relationship between Poynting Vector and Joule's Heating

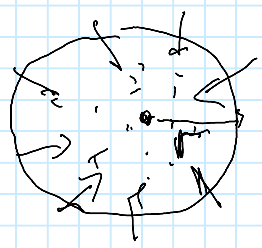


$$\vec{j} = \sigma \vec{E} \;;\quad j = I/A = \frac{I}{\pi r^2} \quad \left(\begin{array}{c}\text{cross section}\\ A = \pi r^2\end{array}\right)$$

$$\oint \vec{B}\, d\vec{l} = \mu_0 I_{enc} \Rightarrow B \cdot 2\pi r = \mu_0 I$$

$$\Rightarrow \boxed{B = \frac{\mu_0 I}{2\pi r}}$$

Poynting Vector: $\boxed{\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0}}$



$$E = \frac{j}{\sigma} = \frac{I}{\sigma\cdot\pi r^2} \Rightarrow \boxed{S = \frac{E\cdot B}{\mu_0} = \frac{E\, I}{2\pi r} = \frac{I^2}{2\pi r\, \sigma\cdot\pi r^2}}$$

The flux of EM energy: $\Phi_W = S\cdot 2\pi r L =$

$$= \frac{I^2 \cdot \cancel{2\pi r} L}{\cancel{2\pi r}\cdot\sigma\cdot\pi r^2} = \frac{I^2\, L}{\sigma\cdot\pi r^2} =$$

$$= I^2\cdot R \quad \text{Joule's heating}$$

$$\boxed{\begin{array}{l} E = \frac{j}{\sigma} = \frac{U}{L} \;; \\ \Rightarrow U = L\frac{j}{\sigma} = \frac{L}{\sigma\cdot\pi r^2} I \end{array}}$$

U - difference of potentials

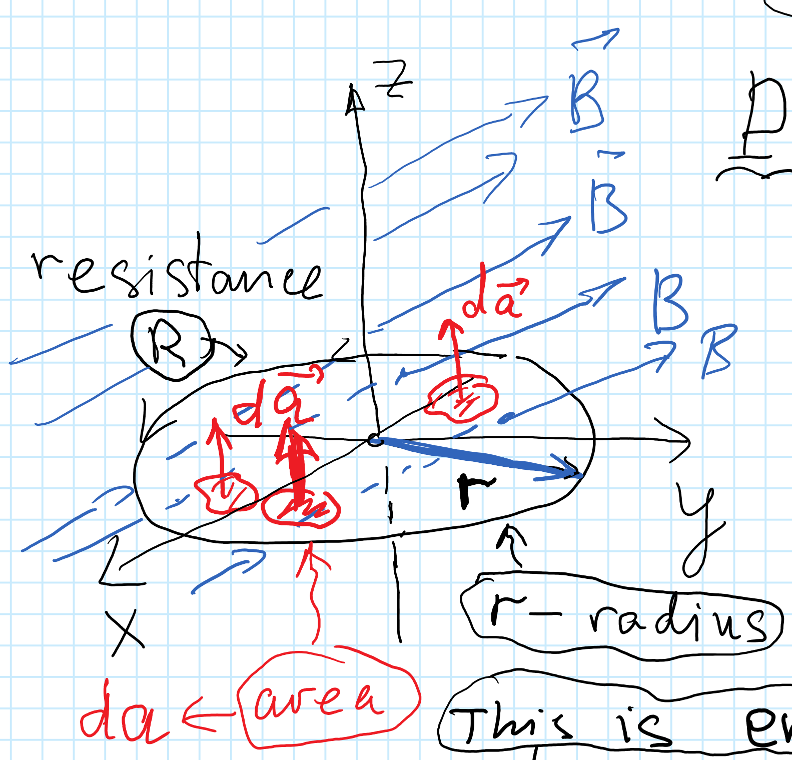
$U = R\, I$

Resistance $\boxed{R = \frac{L}{\sigma\cdot\pi r^2}}$

$$\boxed{\Phi_W = \oint_S \vec{S}\, d\vec{a} = I\cdot U}$$

where $U = RI$

Problem 1 (HW1)



$$\vec{B}(t) = B_0 \frac{\hat{e}_y + \hat{e}_z}{\sqrt{2}}\left(1 - e^{-\mu t}\right)$$

Calculate:

(1) Current $I(t)$?

(2) The $\bar{W}_J(t)$ converted to Joule's heat ?

This is emf

① $\oint_l \vec{E}\, d\vec{l} = -\frac{\partial}{\partial t}\int_S \vec{B}\cdot d\vec{a}$ (area)

$d\vec{a} = \hat{e}_z\cdot da$

$\vec{B}\cdot d\vec{a} = B(t)\cdot da\cdot \hat{e}_z\cdot\frac{\hat{e}_y+\hat{e}_z}{\sqrt{2}} = \frac{B(t)}{\sqrt{2}}da$

$$\oint \vec{E}\, d\vec{l} = I(t)\cdot R = -\frac{\partial}{\partial t}\left(\frac{B(t)}{\sqrt{2}}\pi r^2\right) = +\frac{\mu B_0}{\sqrt{2}} e^{-\mu t}\pi r^2 \Rightarrow \boxed{I(t) = \frac{\mu B_0}{\sqrt{2}\, R}\cdot\pi r^2 e^{-\mu t}}$$

② Joule's heat power

$$dW_J = I^2(t) R\cdot dt$$

$$\bar{W}_J(t) = \int_0^t I^2(t')\cdot R\, dt' = \frac{\mu^2 B_0^2}{2R}\pi^2 r^4 \int_0^t e^{-2\mu t'} dt' \Rightarrow \boxed{\bar{W}_J(t) = \frac{\mu B_0^2 \pi^2 r^4}{4R}\left(1 - e^{-2\mu t}\right)}$$

Integral: $\frac{1}{2\mu}\left(1 - e^{-2\mu t}\right)$

Wave Equations for the Scalar $\varphi(\vec{r},t)$ and Vector $\vec{A}(\vec{r},t)$ Potentials

Griffiths Ch. 10

Maxwell's equations: $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ and $\vec{\nabla}\cdot\vec{B} = 0$

source equations

Dynamic Equations:
$$\begin{cases} \vec{\nabla}\times\vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \Rightarrow \text{from this equation} \\ \vec{\nabla}\times\vec{B} = \mu_0\vec{j} + \mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t} \end{cases}$$

Formulas for the vector-potential $\boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$ and $\boxed{\vec{E} = -\dfrac{\partial\vec{A}}{\partial t} - \vec{\nabla}\varphi}$

From the source equation:

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\left(-\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\varphi\right) = -\frac{\partial}{\partial t}\vec{\nabla}\vec{A} - \vec{\nabla}^2\varphi = \rho/\varepsilon_0$$

$$\Rightarrow \boxed{\vec{\nabla}^2\varphi + \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = -\rho/\varepsilon_0} \quad \text{Eq. 1}$$

$$\frac{\partial\vec{E}}{\partial t}$$

$$\underset{a}{\vec{\nabla}}\times\vec{B} = \vec{\nabla}\times(\underset{b}{\vec{\nabla}}\times\underset{c}{\vec{A}}) = \mu_0\vec{j} + \mu_0\varepsilon_0\left(-\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\frac{\partial\varphi}{\partial t}\right)$$

$$\vec{\nabla}(\vec{\nabla}\vec{A}) - \vec{\nabla}^2\vec{A} = \mu_0\vec{j} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \frac{1}{c^2}\vec{\nabla}\frac{\partial\varphi}{\partial t}$$

The wave equation for $\vec{A}(\vec{r},t)$:

$$\boxed{\vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\cdot\left(\vec{\nabla}\vec{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t}\right) = -\mu_0\vec{j}} \quad \text{Eq. 2}$$

- Equations 1 and 2 describe dynamic changes of the electric and magnetic fields and their interaction with charged particles.

The equations (*) and (**) include terms mixing $\vec{A}$ and φ potentials.