General Solution of the Wave Equation

Lecture 7

02/07/23

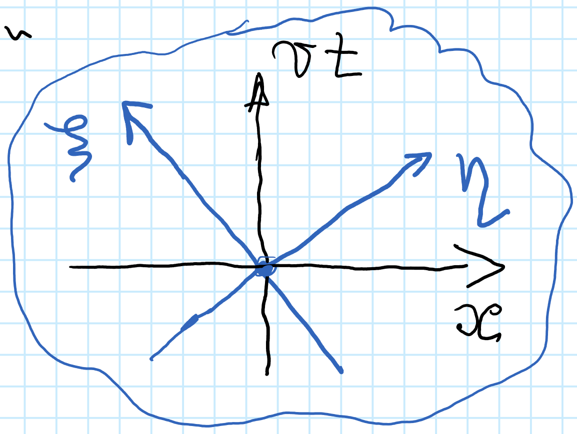
Example of the wave equation for the 1D case:

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f(x,t)}{\partial t^2}$$

$f(x,t) \to$ wave function

New variables:

$$\begin{cases} \eta = x + vt \\ \xi = x - vt \end{cases}$$



Solution of the wave equation: $f(x,t) \equiv f(\eta, \xi)$

The wave equation for new variables η and ξ

$$\begin{cases} \frac{\partial f}{\partial x} = \frac{\partial \eta}{\partial x}\cdot\frac{\partial f}{\partial \eta} + \frac{\partial \xi}{\partial x}\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial \eta} + \frac{\partial f}{\partial \xi} = \left(\frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi}\right) f \\ \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi}\right) f = \left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right)^2 f \end{cases}$$

$$\begin{cases} \frac{\partial f}{\partial t} = \frac{\partial \eta}{\partial t}\frac{\partial f}{\partial \eta} + \frac{\partial \xi}{\partial t}\frac{\partial f}{\partial \xi} = v\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right) f \\ \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\frac{\partial f}{\partial t} = v^2\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right)^2 f \end{cases}$$

$$\boxed{\left(\frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi}\right)^2 f = \left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right)^2 f}$$

$$\cancel{\frac{\partial^2 f}{\partial \eta^2}} + 2\frac{\partial^2 f}{\partial \eta \partial \xi} + \cancel{\frac{\partial^2 f}{\partial \xi^2}} = \cancel{\frac{\partial^2 f}{\partial \eta^2}} - 2\frac{\partial^2 f}{\partial \eta \partial \xi} + \cancel{\frac{\partial^2 f}{\partial \xi^2}} \Rightarrow \boxed{4\frac{\partial^2 f}{\partial \eta \partial \xi} = 0}$$

Solution of the wave equation Eq.(*):

$$f(x,t) = u(\eta) + Z(\xi) = $$
$$= u(x+vt) + Z(x-vt)$$

$u(\eta) \leftarrow$ arbitrary
$Z(\xi) \leftarrow$ functions

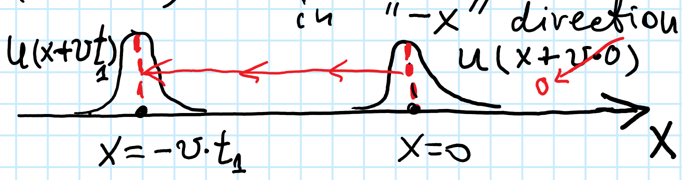
Check:

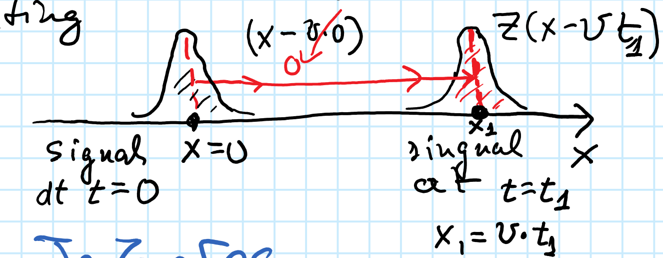
$$\frac{\partial^2 f}{\partial \eta \partial \xi} = \frac{\partial}{\partial \eta}\cdot\frac{\partial}{\partial \xi}\left(u(\eta) + Z(\xi)\right) = $$
$$= \frac{\partial}{\partial \eta}\frac{\partial Z(\xi)}{\partial \xi} = 0$$

Interpretation of these two solutions:

$Z(x-vt) \to$ the wave signal propagating along X-axis.

$u(x+vt) \to$ the wave signal propagating in "$-x$" direction.





Electro Magnetic Waves

Solutions of the wave equations for the EM waves

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The simplest case is the 1D propagation of EM-waves

$$\nabla^2 = \frac{\partial^2}{\partial x^2}$$

$$\frac{\partial^2 \vec{E}}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

General solution for one dimentional space

$$\boxed{\vec{E}(x,t) = \vec{E}_1 \cdot f_+(x-ct) + \vec{E}_2 \cdot f_-(x+ct)}$$

$\vec{E}_1$ and $\vec{E}_2$ are vector constants.

Harmonic waves: harmonic solutions $\to$ $\boxed{\vec{E}(x,t) = e^{-i\omega t}\vec{E}_\omega(x)}$

$$\nabla^2\left(\cancel{e^{-i\omega t}}\vec{E}_\omega(x)\right) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[e^{-i\omega t}\cdot\vec{E}_\omega(x)\right] = -\frac{\omega^2}{c^2}\cancel{e^{-i\omega t}}\vec{E}_\omega(x)$$

$$\nabla^2 \vec{E}_\omega(x) = \frac{d^2}{dx^2}\vec{E}_\omega(x) = -\frac{\omega^2}{c^2}\vec{E}_\omega(x) \quad \boxed{k = \frac{\omega}{c}} \to \text{the wave number}$$

$$\vec{E}''_\omega(x) + k^2\vec{E}_\omega(x) = 0 \leftarrow \text{Helmholtz equation}$$

Solutions of this equation

Two independent solutions: $\sin(kx)$, $\cos(kx)$ $\Rightarrow$ $\boxed{\vec{E}_\omega(x) = \vec{E}_1 e^{+ikx} + \vec{E}_2 e^{-ikx}}$

(We should take real or imaginary parts: "$\sin kx$" or "$\cos kx$")

(1) $\vec{E}_1(x,t) = e^{-i\omega t}\vec{E}_1 e^{ikx} = \vec{E}_1 \cdot e^{-i(\omega t - kx)}$

(2) $\vec{E}_2(x,t) = \vec{E}_2 \cdot e^{-i(kx+\omega t)}$

$\frac{\omega}{k} = c$

Conclusions: $\vec{E}_1(x,t) = \vec{E}_1 e^{-ik(\frac{\omega}{k}t - x)} = \vec{E}_1 e^{ik(x-ct)} \leftarrow$ wave propagating in the "x"-direction

$\vec{E}_2(x,t) = \vec{E}_2 e^{-ik(\frac{\omega}{k}t + x)} = \vec{E}_2 e^{-ik(x+ct)} \leftarrow$ wave propagating in the "$-x$" direction

Magnetic field:

$$\nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} \Rightarrow \boxed{\vec{B}(x,t) = \vec{B}_1 e^{i(kx-\omega t)} + \vec{B}_2 \cdot e^{-i(kx+\omega t)}}$$

harmonic waves

$\vec{B}_1$ and $\vec{B}_2$ - amplitudes

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Leftarrow \begin{cases} \vec{E} = \vec{E}_0 \cdot e^{i(\vec{k}\vec{r}-\omega t)} \\ \vec{B} = \vec{B}_0 \cdot e^{-i(\vec{k}\vec{r}-\omega t)} \end{cases}$$

3D-plane EM waves

$\vec{k} = \frac{\omega}{c}\cdot \hat{e}_k$

wave vector

relation between $\vec{E}_0$ and $\vec{B}_0$?

$\phi = \vec{k}\vec{r} - \omega t$

$\vec{k}\vec{r} = \text{const.}$ on the plane (S)

$$\nabla \times \vec{E} = \nabla \times (\vec{E}_0 \cdot e^{i(\vec{k}\vec{r}-\omega t)}) =$$

$$= e^{-i\omega t}\left(\nabla e^{i\vec{k}\vec{r}} \times \vec{E}_0\right) =$$

$$= e^{-i\omega t}(i\vec{k} \times \vec{E}_0)\cdot e^{i\vec{k}\vec{r}} = i(\vec{k}\times\vec{E}_0)e^{i(\vec{k}\vec{r}-\omega t)}$$

$$i\vec{k}\times\vec{E}_0 e^{i(\vec{k}\vec{r}-\omega t)} = -\frac{\partial \vec{B}}{\partial t} = +i\omega \vec{B}_0 \cdot e^{i(\vec{k}\vec{r}-\omega t)}$$

$$\vec{E}_0, \vec{B}_0 \perp \vec{k}$$

$$\Rightarrow \begin{cases} \vec{B}_0 = \frac{\vec{k}}{\omega}\times\vec{E}_0 = \frac{1}{c}(\hat{e}_k \times \vec{E}_0) \\ \omega = ck \qquad \vec{B}_0 \perp \vec{E}_0 \end{cases}$$

Amplitudes $\boxed{E_0 = c\cdot B_0}$

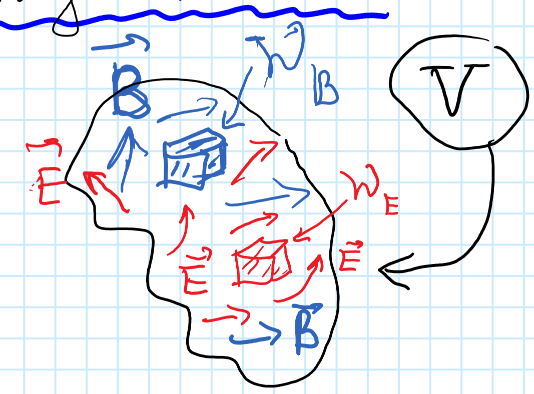
The vector form:

$$\boxed{\vec{E} = c(\vec{B}\times\hat{e}_k)}$$

Electromagnetic Energy Conservation and Poynting Vector

Electromagnetic (EM) Energy:

$$W = W_E + W_B = \underbrace{\int_V \frac{\varepsilon_0 \vec{E}^2}{2}\cdot d^3r}_{W_E} + \underbrace{\int_V \frac{\vec{B}^2}{2\mu_0}\cdot d^3r}_{W_B}$$



the volume density of the EM energy:

$$\boxed{\begin{aligned} w_E &= \frac{\varepsilon_0 \vec{E}^2}{2} \\ w_B &= \frac{B^2}{2\mu_0} \end{aligned}}$$

$W = \int_V (w_E + w_B) d^3r \equiv \int_V w\, d^3r$

where $\boxed{w = w_E + w_B = \frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}}$

is the volume density of the EM energy.

These equations are valid for the static field and for EM wave.

Maxwell's Equations describes dynamical transformations of the EM energy for all electric and magnetic phenomena.

Dynamical equations for the EM field:

Left and right parts scalar product

$$\begin{cases} \vec{B}\cdot\ldots \quad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \\ \vec{E}\cdot\ldots \quad \nabla\times\vec{B} = \mu_0\vec{j} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \end{cases}$$

$$\vec{B}(\nabla\times\vec{E}) = -\vec{B}\frac{\partial\vec{B}}{\partial t}$$

$$\vec{E}(\nabla\times\vec{B}) = \mu_0\vec{j}\vec{E} + \mu_0\varepsilon_0\vec{E}\frac{\partial\vec{E}}{\partial t}$$

$$\vec{B}\frac{\partial\vec{B}}{\partial t} = \frac{1}{2}\frac{\partial\vec{B}^2}{\partial t}; \quad \vec{E}\frac{\partial\vec{E}}{\partial t} = \frac{1}{2}\frac{\partial\vec{E}^2}{\partial t}$$

$$\boxed{\vec{E}\cdot(\nabla\times\vec{B}) - \vec{B}\cdot(\nabla\times\vec{E}) = \mu_0\frac{\partial}{\partial t}\left(\frac{\varepsilon_0\vec{E}^2}{2} + \frac{B^2}{2\mu_0}\right) + \mu_0\vec{j}\vec{E}} \quad \text{Eq.}(*)$$

This term is the volume density of electric and magnetic energies!

Vector algebra:

$\vec{a}(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\vec{c} = (\vec{c}\times\vec{a})\cdot\vec{b}$

$\nabla\cdot(\vec{E}\times\vec{B}) = \vec{B}(\nabla\times\vec{E}) - (\nabla\times\vec{B})\cdot\vec{E} \Rightarrow$

The left part of Eq.(*): it is the term

$$-\nabla\cdot(\vec{E}\times\vec{B})$$

Eq.(*) can be written as:

$$\frac{\partial}{\partial t}\underbrace{\left(\frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}\right)}_{w} + \nabla\cdot\underbrace{\left(\frac{\vec{E}\times\vec{B}}{\mu_0}\right)}_{S} = -\vec{j}\cdot\vec{E}$$

$$\boxed{\nabla\vec{S} + \frac{\partial w}{\partial t} = -\vec{j}\cdot\vec{E}}$$

$$\boxed{\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0}}$$

Poynting vector