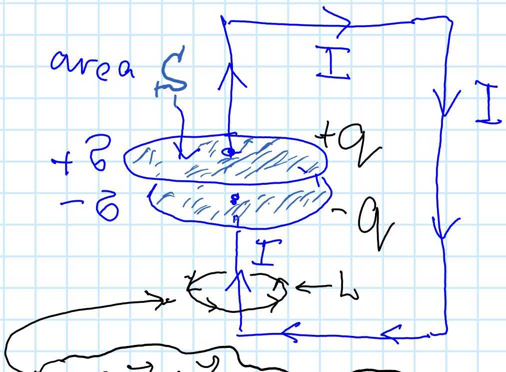
Lecture 7

02/02/2023

Displacement Current in Maxwell's Equations



$\oint_L \vec{B}d\vec{l} = \mu_0 I_{enc}$

Qualitative consideration of the displacement current:

Maxwell's equation for the magnetostatic field:

$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{j}}$ or the integral form $\boxed{\oint_L \vec{B}d\vec{l} = \mu_0 I_{enc}}$

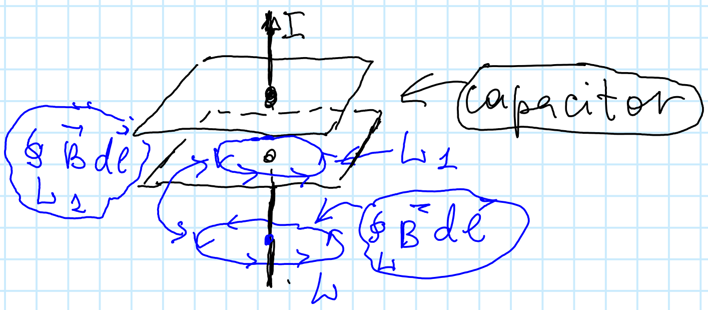
The electric current $I(t)$ is the current of the capacitor discharge:

$\sigma(t)$ is the surface charge density

$$I(t) = -\frac{dq}{dt} = -\frac{d(S\cdot\sigma(t))}{dt} = -S\,\frac{d\sigma}{dt}$$

The magnetic field around a wire, which is "outside" the capacitor:

$$\boxed{\oint_L \vec{B}d\vec{l} = \mu_0 I_{enc} = -\mu_0\frac{d}{dt}[S\cdot\sigma(t)] = -\mu_0 S\frac{d\sigma}{dt}}$$

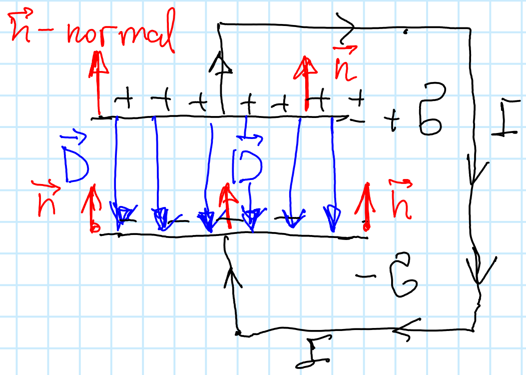


Magnetic field in areas, which are close to wire, can be aproximately computed as

$$\boxed{\begin{gathered}\oint_L \vec{B}\cdot d\vec{l} \simeq B\cdot 2\pi r = \mu_0 I_{enc}\\ B \simeq \frac{\mu_0 I_{enc}}{2\pi r}\end{gathered}}$$

We can calculate circulation of the $\vec{B}$-field inside the capacitor:

$\oint_{L_1} \vec{B}d\vec{l}$



The circulation of the B-vector inside the capacitor has to be similar or equal to the value of circulation $\oint_L \vec{B}d\vec{l}$ outside the capacitor.

The reason for this: $\vec{B}$-field has to be continuous and close regions "inside" and "outside" capacitor should have the same $\vec{B}$. Inside the capacitor: $I=0$, but the variable electric field $E(t) = \sigma/\varepsilon_0$ relates to the capacitor charge.

$$\boxed{\oint_{L_1} \vec{B}d\vec{l} = -\mu_0\frac{dq}{dt} = -\mu_0 S\frac{d\sigma}{dt} = -\mu_0 S\frac{d}{dt}(\varepsilon_0 E) = +\mu_0\frac{\partial}{\partial t}(\vec{S}\vec{D})}$$

Charge density $\sigma(t)$:

$$\boxed{\sigma(t) = \varepsilon_0 E(t) = D(t)}$$

The displacement vector $\vec{D}$:

The normal vector $\vec{n}$ is perpendicular to the capacitor plates and is oriented along the electric current direction. The value "$\vec{S}\vec{D}$" is the flux of the electric displacement vector $\vec{D}$.

$$\boxed{\vec{D} = -\sigma(t)\vec{n}}$$

The $\vec{B}$-vector circulation inside the capacitor:

$\boxed{\oint_{L_1} \vec{B}d\vec{l} = \mu_0\frac{\partial}{\partial t}N_D}$ where $\boxed{N_D = \int_S \vec{D}\cdot d\vec{S}}$ flux of the $\vec{D}$-vector

The formula for the $\vec{B}$- circulation outside and inside the capacitor can be written as:

$$\boxed{\oint_L \vec{B}\,d\vec{l} = \begin{cases} \mu_0 \int_S \vec{j}\,d\vec{S}, & \text{outside capacitor} \\ \mu_0 \int_S \frac{\partial \vec{D}}{\partial t} d\vec{S}, & \text{inside capacitor} \end{cases}} \quad (*)$$

In our example: $\dot{\vec{D}} = \vec{D} = 0$ and $\vec{j} \neq 0$ outside capacitor

$\dot{\vec{D}} \neq 0$ and $\vec{j} = 0$ inside capacitor

More general case: $\vec{j} \neq 0$ and $\dot{\vec{D}} \neq 0$ in the same regions.

From Eq. (*) $\boxed{\oint_L \vec{B}\,d\vec{l} = \mu_0 \int_S \left(\vec{j} + \frac{\partial \vec{D}}{\partial t}\right) d\vec{S}}$ $\leftarrow$ $\vec{D} = \varepsilon_0 \vec{E}$

Continuity equation $\rightarrow$ $\boxed{\vec{\nabla}\vec{j} + \frac{\partial \rho}{\partial t} = 0}$

Maxwell's equation:

$\vec{\nabla}\vec{E} = \rho/\varepsilon_0 \Rightarrow$

$\rho = \vec{\nabla}(\varepsilon_0 \vec{E}) = \vec{\nabla}\vec{D}$

We can introduce the density $\vec{j}_{total}$ of the total current:

$\vec{\nabla}\cdot\vec{j} + \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{D} = 0$

$\boxed{\vec{\nabla}\left(\vec{j} + \frac{\partial}{\partial t}\vec{D}\right) = 0}$

$\boxed{\vec{\nabla}\cdot\vec{j}_{total} = 0}$

$\boxed{\vec{j}_{tot} = \vec{j} + \frac{\partial \vec{D}}{\partial t}}$

the current of charged particles

the displacement current

Circulation of the $\vec{B}$-fied:

$$\boxed{\oint_L \vec{B}\,d\vec{l} = \mu_0 \int_S \vec{j}_{total}\cdot d\vec{S}}$$

This is the experimental law in different forms.

$\rightarrow$ integral form

$\rightarrow$ differential form

Maxwell's equation:

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0 \vec{j}_{total} = \mu_0\left(\vec{j} + \frac{\partial \vec{D}}{\partial t}\right)} \quad \text{Eq.}(**)$$

$\frac{\partial \vec{D}}{\partial t} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ – the density of the displacement current

Time-dependent Maxwell's Equations

(1) $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ (Coulomb's Law)

(2) $\vec{\nabla}\cdot\vec{B} = 0$ ("no magnetic charges")

(3) $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Faraday's Law)

(4) $\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{j} + \varepsilon_0\frac{\partial \vec{E}}{\partial t}\right)$ (Biot-Savart Law and displacement current)

$\rho = \rho(\vec{r}, t)$

$\vec{j} = \vec{j}(\vec{r}, t)$

Electromagnetic wave equation in vacuum.

$\boxed{\vec{j} = 0 \,;\ \vec{\rho} = 0}$

$$\begin{cases} (*) & \vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ (**) & \vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t} \end{cases}$$

New notation:

$$\boxed{c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}}$$

$\boxed{c \text{ - speed of light}}$

$$\boxed{\underset{a}{\vec{\nabla}} \times (\underset{b}{\vec{\nabla}} \times \underset{c}{\vec{E}}) = -\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})}$$

from Eq. (**)

$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a}\vec{c}) - \vec{c}(\vec{a}\vec{b})$

$$\vec{\nabla}(\cancelto{0}{\vec{\nabla}\vec{E}}) - \vec{\nabla}^2 E = -\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\mu_0 \varepsilon_0 = \frac{1}{c^2}$$

The wave equation for the $\vec{E}$-field:

$$\boxed{\vec{\nabla}^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}}$$

The wave equation for the $\vec{B}$-field:

$$\boxed{\vec{\nabla}^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}}$$