Lecture 5

1/31/2023

recap:

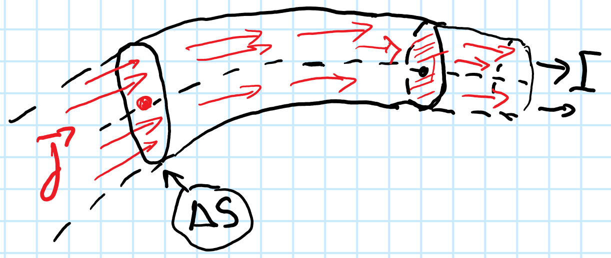
The energy of magnetic field:

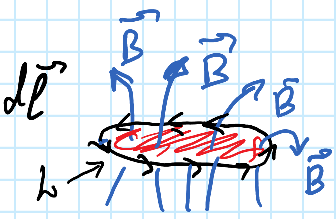
$$W_B = \frac{1}{2}\Phi \cdot I \equiv \frac{1}{2}\oint_L \vec{A}\,d\vec{\ell}\cdot I = \frac{1}{2}\int \vec{A}\cdot j\cdot \underbrace{\Delta S\cdot d\ell}_{d^3r}\cdot\hat{e}_j = \frac{1}{2}\int \vec{j}(\vec{r},t)\,\vec{A}(\vec{r},t)\,d^3r$$

I

$\hat{e}_j \equiv \hat{e}_v \quad \vec{j} = j\cdot\hat{e}_v$

$$\Phi = \int_S \vec{B}\,d\vec{s} = \oint_L \vec{A}\cdot d\vec{\ell}$$





$$\boxed{W_B = \frac{1}{2}\int \vec{j}(\vec{r},t)\,\vec{A}(\vec{r},t)\,d^3r}$$

Density of Magnetic Energy

The energy of electric current in magnetic field is given by the formula:

$$\boxed{W = \frac{1}{2}\int \vec{j}(\vec{r},t)\,\vec{A}(\vec{r},t)\,d^3r}$$

$\vec{A}(\vec{r},t)$ is the vector potential

We can express the energy via the value of the magnetic field $\vec{B}$:

$$\boxed{\vec{B} = \vec{\nabla}\times\vec{A}} \quad \text{and} \quad \boxed{\vec{j} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B}}$$

Maxwell equation

$$\boxed{W = \frac{1}{2\mu_0}\int (\vec{\nabla}\times\vec{B})\cdot\vec{A}\,d^3r}$$

From the vector algebra:

$$(\vec{a}\times\vec{b})\cdot\vec{c} = (\vec{c}\times\vec{b})\cdot\vec{a} = (\vec{a}\times\vec{c})\cdot\vec{b}$$

$$\vec{\nabla}(\vec{a}\times\vec{b}) = (\vec{\nabla}\times\underset{\vec{B}}{\vec{a}})\cdot\underset{\vec{A}}{\vec{b}} - (\vec{\nabla}\times\vec{b})\cdot\vec{a} \qquad (\vec{\nabla}_r\text{ - operator, acting on } \vec{a} \text{ and } \vec{b})$$

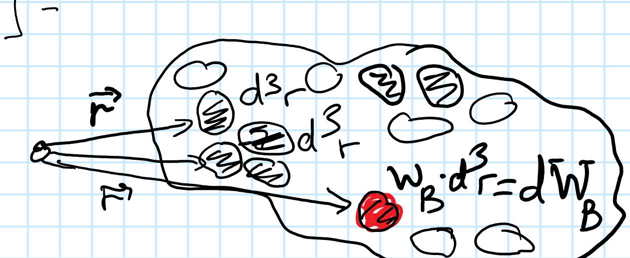
The mixed product

$$(\vec{\nabla}\times\vec{B})\cdot\vec{A} = \vec{\nabla}(\vec{B}\times\vec{A}) + \underbrace{(\vec{\nabla}\times\vec{A})}_{\vec{B}}\cdot\vec{B} = \vec{\nabla}(\vec{B}\times\vec{A}) + \vec{B}^2$$

The energy of magnetic field and the volume density of energy w_B

$$W = \frac{1}{2\mu_0}\int_V (\vec{\nabla}\times\vec{B})\vec{A}\,d^3r = \frac{1}{2\mu_0}\left[\int_V \vec{\nabla}(\vec{B}\times\vec{A})\,d^3r + \int_V \vec{B}^2\,d^3r\right] =$$

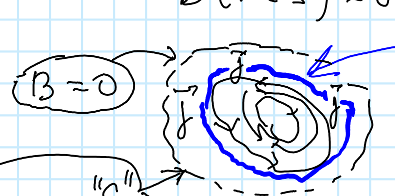
$$= \frac{1}{2\mu_0}\left[\oint_S \underbrace{(\vec{B}\times\vec{A})\,d\vec{S}}_{\to 0} + \int \vec{B}^2\,d^3r\right].$$



$$\boxed{W_B = \int_V dW_B = \int w_B\,d^3r}$$

On the boundary "S" of our system $\vec{B}$ and $\vec{A}$ have to be zero. ("S" could be infinitely large)

$\vec{B}(\vec{r}\in S) = 0$ and $\vec{A}(\vec{r}\in S) = 0$



"S_j" Boundary of the current region $\vec{j}\neq 0$

The boundary surface for $\vec{B}$ and $\vec{A}$

The final formula:

$$\boxed{W_B = \frac{1}{2\mu_0}\int \vec{B}^2\,d^3r}$$

The value "$\vec{B}^2/2\mu_0$" is called the magnetic energy density $w_B = \frac{B^2}{2\mu_0}$

This is the energy of magnetic field per unit of space volume. The energy density is different at different locations. The total energy of magnetic field is:

$$\boxed{W_B = \int w_B(\vec{r})\,d^3r = \int \frac{\vec{B}^2(\vec{r})}{2\mu_0}\,d^3r}$$

where $w_B = B^2/2\mu_0$.

Summary: Energy of Magnetic Field

These equations can be used for solutions of ED problems.

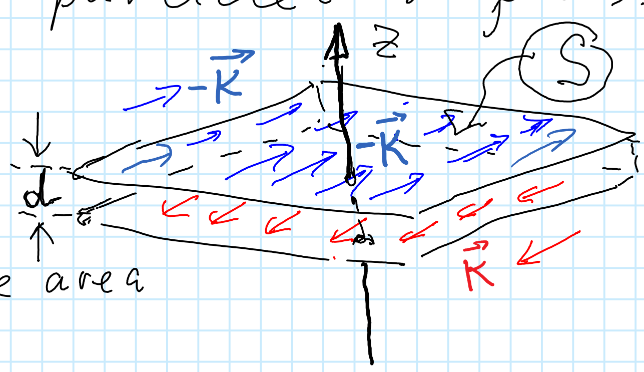
$$\boxed{W_B = \int_V \frac{B^2}{2\mu_0}\, d^3r'} \qquad \boxed{w_B = \frac{B^2(\vec{r})}{2\mu_0}} \qquad \boxed{W_B = \int_V w_B(\vec{r}', t)\, d^3r'}$$

Example I:

Energy of magnetic field

Uniform surface currents with the surface densities $\vec{K}$ and $-\vec{K}$ flowing over two parallel surfaces.

The distance between two surfaces is "d". Calculate the magnetic energy W_B of the system, if the area of each surface is S and $d \ll \sqrt{S}$.



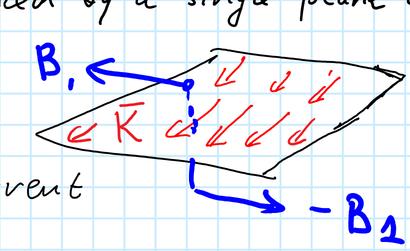
Solution:

The magnetic field exists only between these surfaces. The B_1 field induced by a single plane current $\vec{K}$

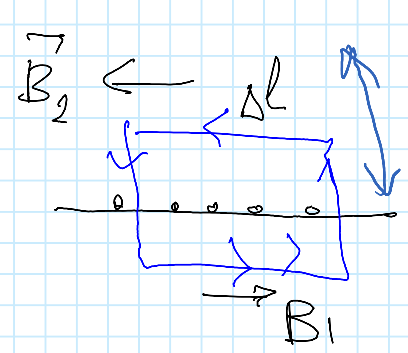
$$\boxed{\oint_L \vec{B}\,d\vec{l} = \mu_0 I_{inc}}$$

$$\oint_{\Delta l} \vec{B}_1 d\vec{l} = B_1 \Delta l + B_1 \Delta l = \mu_0 \underbrace{\Delta l \cdot K}$$

$$\boxed{B_1 = \mu_0 K/2}$$ enclosed current



The magnetic field B induced by two surfaces:



$$\begin{cases} B = 2B_1 = \mu_0 K & \text{- in the layer between surfaces} \\ B = 0 & \text{- in the entire space out of layer} \end{cases}$$

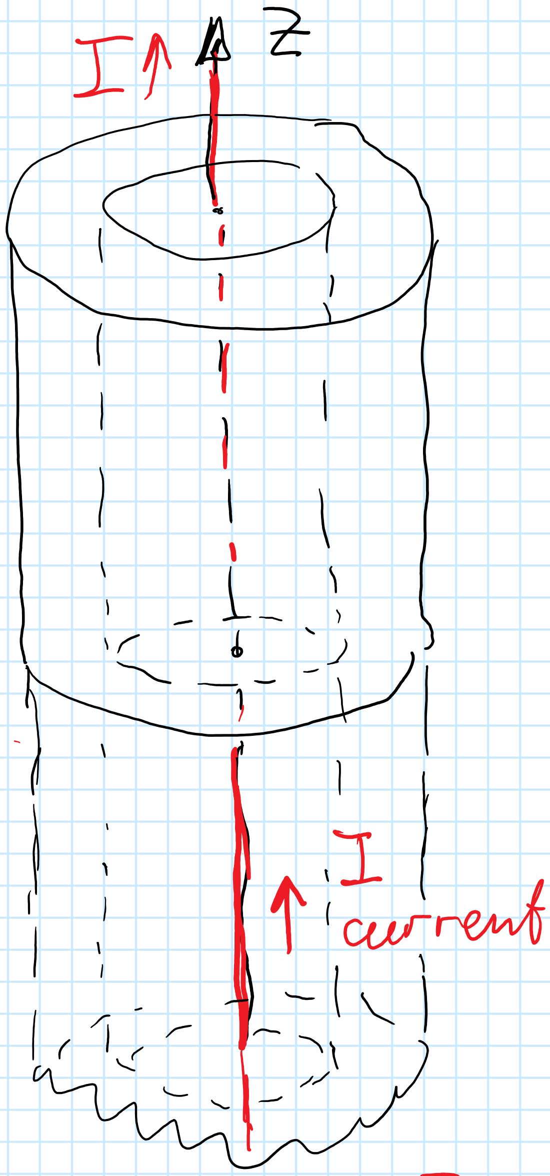
The density of magnetic field energy w_B:

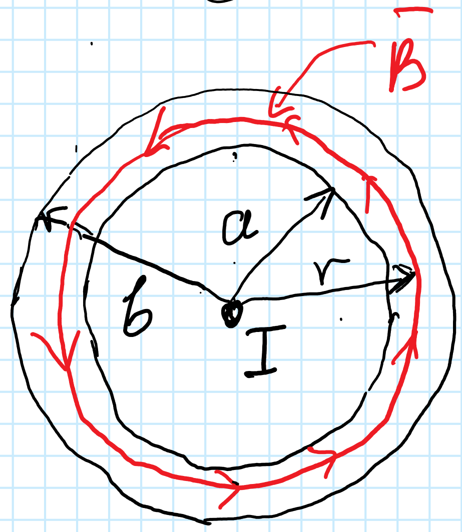
$$w_B = \frac{B^2}{2\mu_0} = \frac{\mu_0^2 K^2}{2\mu_0} = \frac{\mu_0 K^2}{2}$$

The magnetic energy W_B of the considered system is:

$$W_B = \int_V w_B\, d^3r = \frac{\mu_0 K^2}{2} \cdot \underbrace{S \cdot d}_{\text{volume}}$$

Example II: Calculate the energy of magnetic field per unit of length of infinitely long cylindrical layer. Magnetic field $\vec{B}$ is induced by the electric current I propagating along the cylinders' axis z. The layer inner and outer radii are $r_i = a$ and $r_o = b$ respectively.





Solution

(1) Determine the magnetic field $\vec{B}(\bar{r})$ induced by the infinite straight current I.

$$\oint_L \vec{B}\, d\vec{\ell} = B(r)\cdot 2\pi r = \mu_0 I_{enc} = \mu_0 I$$

$$\Rightarrow \quad B = \frac{\mu_0 I}{2\pi r}$$

(2) The density of the magnetic field energy w_B:

$$w_B(r) = \frac{B^2(r)}{2\mu_0} = \frac{\mu_0 I^2}{8\pi^2 r^2}$$

(3) The total energy of magnetic field W_B:

The cylinder length $L = 1$ $\Rightarrow$

$$W_B = \int_V w_B(r)\cdot d^3 r = \int_{r=a}^{r=b} w_B(r)\cdot 2\pi r\, dr$$

$$W_B = \frac{\mu_0 I^2}{4\pi}\int_a^b \frac{dr}{r} = \frac{\mu_0 I^2}{8\pi} \ln r \Big|_a^b$$

The total energy: $\boxed{W_B = \frac{\mu_0 I^2}{8\pi} \ln\left(\frac{b}{a}\right)}$