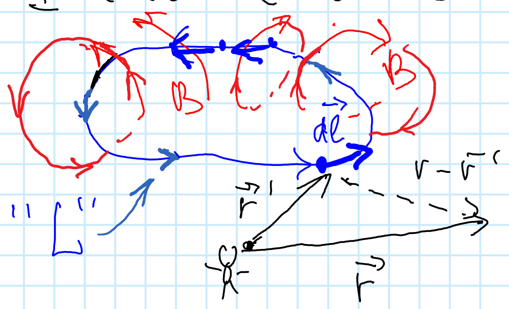
Lecture 4

01/26/23

Mutual Inductance

Griffiths 7.2

I-electric current



("Biot-Savart Law" - information from previous semester)

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_L \frac{d\vec{\ell}\times\hat{e}_{rr'}}{|\vec{r}-\vec{r}'|^2} \equiv \frac{\mu_0}{4\pi}\int_V \frac{\vec{j}(\vec{r}')\times\hat{e}_{rr'}\,d^3r'}{|\vec{r}-\vec{r}'|^2} \Leftrightarrow \vec{\nabla}\times\vec{B} = -\mu_0\vec{j}$$

$d\vec{\ell} \equiv d\vec{r}'$

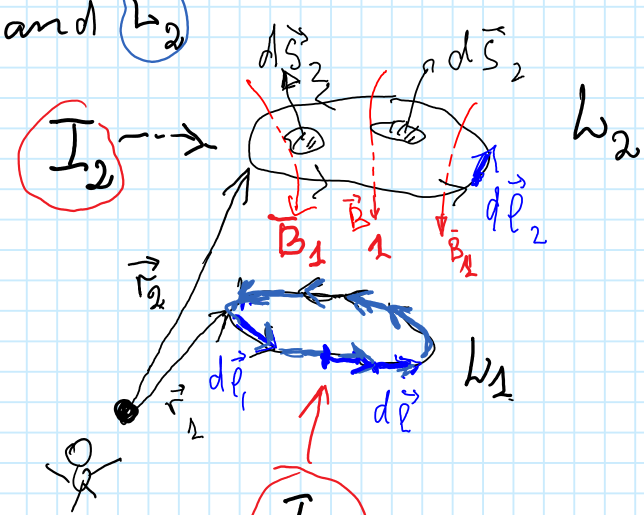
$\vec{j}(\vec{r})$ – the density of electric current

$$\boxed{\hat{e}_{rr'} = \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}}$$

Solution of Poisson's equation $\vec{\nabla}^2\vec{A} = -\mu_0\vec{j}$:

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{j}(\vec{r}')d^3r'}{|\vec{r}-\vec{r}'|}}$$

Wire loops L_1 and L_2



Electric current

We can calculate magnetic flux Φ_{21} through the loop "2" induced by the loop "1":

$$\Phi_{21} = \int_{S_2}\vec{B}_1 d\vec{S}_2 = \oint_{L_2}\vec{A}_1 d\vec{\ell}_2 = \frac{\mu_0 I_1}{4\pi}\oint_{L_2}\oint_{L_1}\frac{d\vec{\ell}_1\cdot d\vec{\ell}_2}{|\vec{r}_2-\vec{r}_1|}$$

because $\vec{A}_1(\vec{r}) = \frac{\mu_0 I_1}{4\pi}\oint_{L_1}\frac{d\vec{\ell}_1}{|\vec{r}-\vec{r}_1|}$ $\quad \begin{cases}\vec{r}_2 \in L_2\\ \vec{r}_1 \in L_1\end{cases}$

The final formula for the flux Φ_{21}:

$$\Phi_{21} = \left(\frac{\mu_0}{4\pi}\oint_{L_1}\oint_{L_2}\frac{d\vec{\ell}_1 d\vec{\ell}_2}{|\vec{r}_2-\vec{r}_1|}\right)\cdot I_1 = M_{21}\cdot I_1, \text{ where } \boxed{M_{21} = \frac{\mu_0}{4\pi}\oint_{L_1}\oint_{L_2}\frac{d\vec{\ell}_1 d\vec{\ell}_2}{|\vec{r}_2-\vec{r}_1|}}$$

mutual inductance

$$\boxed{\Phi_{21} = M_{21}\cdot I_1}$$

M_{21} = Constant

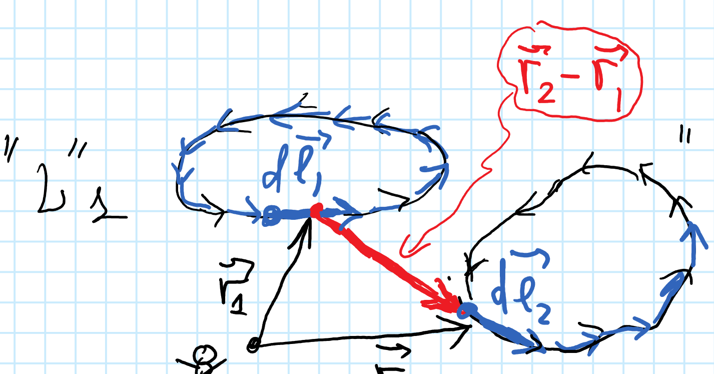
↳ depends on the loop geometry

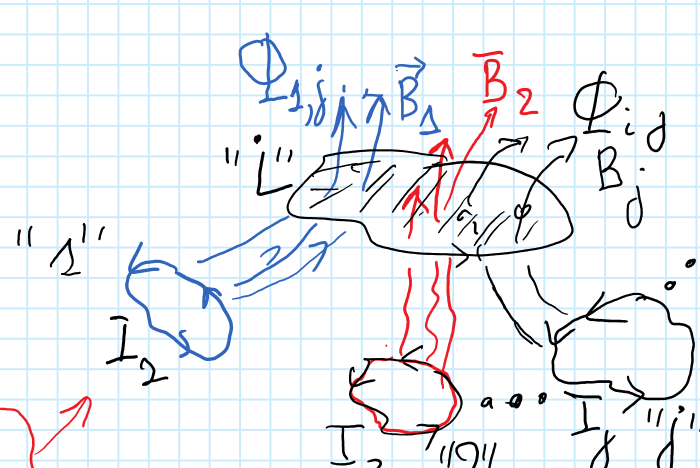
It is not difficult to show:

$$\boxed{\Phi_{12} = M_{12}\cdot I_2} \text{ and } \boxed{M_{12} = \frac{\mu_0}{4\pi}\oint_{L_1}\oint_{L_2}\frac{d\vec{\ell}_2\cdot d\vec{\ell}_1}{|\vec{r}_1-\vec{r}_2|} = M_{21}} \qquad \boxed{M_{21} = M_{12}}$$

Coefficients of mutual inductance (Summary)

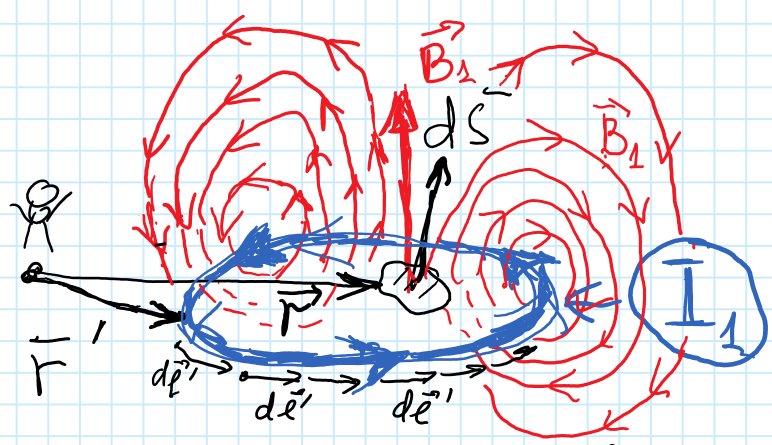
$$M_{21} = M_{12} = \frac{\mu_0}{4\pi}\oint_{L_1}\oint_{L_2}\frac{d\vec{\ell}_1\cdot d\vec{\ell}_2}{|\vec{r}_2-\vec{r}_1|} \qquad \boxed{M_{i,j} = \frac{\Phi_{ij}}{I_j}}$$





$$\boxed{\Phi_i = \sum_k \Phi_{ik} = \sum_k M_{ik} I_k}$$

Self-induction:



(cross section ΔS; $\Delta S \to 0$)

Biot-Savart Law

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{j}_1(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,d^3r'}$$

$$\boxed{\vec{j}_1(\vec{r}) = \frac{I_1}{\Delta S}}$$

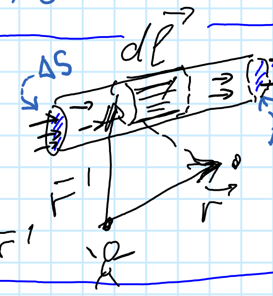
$\vec{j}_1 \rightleftarrows I_1$

$$\vec{B} = \frac{\mu_0 I_1}{4\pi}\oint_L \frac{d\vec{\ell}\,'\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

outside wire

$\vec{r}\,' \in L$

$$dB = \frac{\mu_0 I_1}{4\pi}\frac{d\ell'\cdot \sin\alpha}{|\vec{r}-\vec{r}\,'|^2}$$



$dB \to \infty$, if $\vec{r} \to \vec{r}\,'$

$B_1 \propto I_1$

and $\Phi_{11} = \int_{S_1} \vec{B}_1 d\vec{s}_1 = M_{11} I_1$; $\Rightarrow$ $\boxed{\Phi_{11} = L_1 I_1}$

$\Phi_{11} \propto I_1$

$L_1 = M_{11}$ ← self-inductance

the coefficient of self-induction

Units $[M_{21}] = [L] = H \equiv \frac{\text{Volt}\cdot\text{second}}{\text{Amper}}$

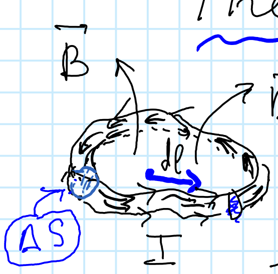
henry

Faraday's Law for the self-induction:

$$\boxed{\mathcal{E}mf = \oint \vec{E}\,d\vec{\ell} = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}}$$

Home Reading: Griffiths, Section 7.2.3 "Solenoid"

The energy W_H of an electric current in the magnetic field B



$$\mathcal{E}mf = \oint \vec{E}\,d\vec{\ell} = -\frac{d\phi}{dt}$$

The work of the electric field for the time-interval dt calculated for a single charge particle q: $dW_q^{(1)} = q\,\vec{E}\cdot d\vec{\ell} = q\vec{E}\vec{v}\,dt$ ← $\boxed{v\,dt = d\ell}$

$\boxed{I = nqv\cdot\Delta S}$

density, velocity, particle charge

The total work on all particles around the loop:

$$dW_q = \oint_\ell \underbrace{d\ell\,\Delta S}_{dV}\cdot n\cdot dW_q^{(1)} = dt\cdot q\cdot\Delta S\, n\oint_\ell d\ell\,\vec{E}\cdot\vec{v} = dt\,\underbrace{(qvn\Delta S)}_{\text{electric current } I}\oint_\ell d\vec{\ell}\,\vec{E}\,;$$

n – particle volume density

$\begin{cases}\vec{v} = v\,\vec{e}_v\,;\\ \vec{e}_v = \vec{e}_\ell\end{cases}$

The charge flux: $dQ = I\cdot dt$

The work dW_q is done due to spending of the energy of magnetic field dW_B:

$$dW_B = -dW_q = -dt\cdot I\cdot\oint \vec{E}\,d\vec{\ell}$$

$$\underbrace{dW_q}_{\text{work}} = -\underbrace{dW_B}_{\text{field energy}}$$

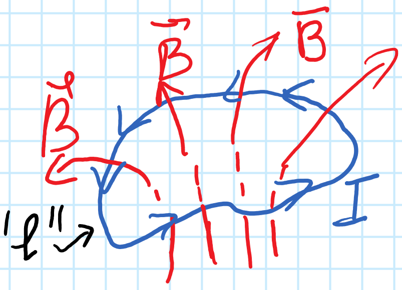
$$dW_B = +dt\cdot I\,\frac{d\Phi}{dt} = d(\Phi I)$$

$$\Rightarrow \boxed{W_B = \Phi\cdot I}$$

This is the energy of the curren I in the external magnetic field B

Summery

The energy of a loop with the electric current I in the external magnetic field B.



$$\boxed{W_B = \Phi \cdot I} \quad \text{where} \quad \Phi = \int_S \vec{B}\,d\vec{s} = \oint_\ell \vec{A}\,d\vec{\ell}$$

Self-Induction and self-energy of a loop with the electric current

Faraday's Law of EM induction:

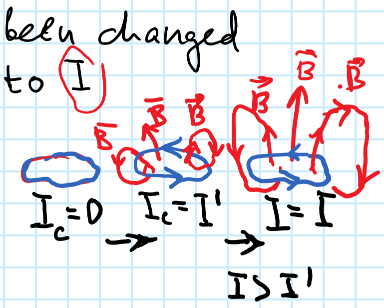
$$\boxed{\oint_\ell \vec{E}\,d\vec{\ell} = -\frac{\partial}{\partial t}\left(\int_S \vec{B}\,d\vec{s}\right) = -\frac{\partial}{\partial t}\left(\oint_\ell \vec{A}\,d\vec{\ell}\right)}$$

Self-induction: $\Phi = L I$

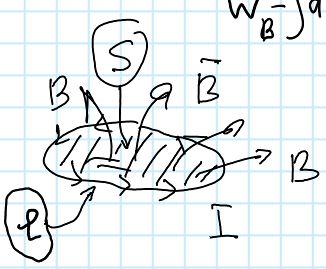
From Faraday's Law: $\oint \vec{E}\,d\vec{\ell} = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}$

$dW_B = \Phi \cdot dI' = L \cdot I' \cdot dI'$

Current has been changed from 0 to I



$$W_B = \int dW_B = L \cdot \int_0^I I'\,dI' \Rightarrow \boxed{W_B = \frac{LI^2}{2}}$$



or $\boxed{W_B = \frac{1}{2} I \cdot \Phi}$

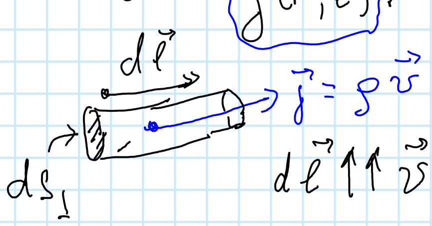
$$\Phi = \int_S \vec{B}\,d\vec{s} = \oint_\ell \vec{A}\,d\vec{\ell}$$

or $\boxed{W_B = \frac{1}{2}\oint_\ell \vec{A} \cdot I\,d\vec{\ell}}$

the energy of magnetic field

We can extend these formulas for continuosly distributed current $\vec{j}(\vec{r},t)$:

$I = \int \vec{j} \cdot d\vec{s} \Rightarrow W_B = \frac{1}{2}\oint j\,ds_\perp\,d\vec{\ell} \cdot \vec{A} \Rightarrow$



$$\Rightarrow \boxed{W_B = \frac{1}{2}\int_V \vec{j} \cdot \vec{A}\,d^3r}$$

$j\,d\vec{\ell} = \vec{j} \cdot d\ell$ and $ds_\perp d\ell = d^3r$

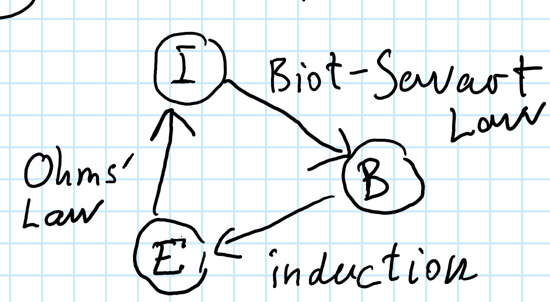
$\vec{j} \cdot d\vec{s} = j \cdot ds_\perp$

Self-action of electric current:

I induced $B \to$ acts via $\oint \vec{E}\,d\vec{\ell}$ on the current I

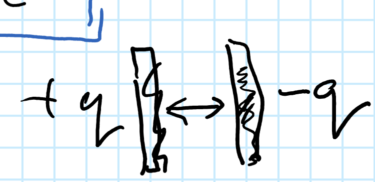
The self-energy of the current I in the magnetic field $\vec{B}$ induced by the same current.

$$\boxed{W_B = \frac{1}{2} L I^2 = \frac{\Phi^2}{2L}}$$



$$\boxed{W_E = \frac{1}{2} C U^2 = \frac{q^2}{2C}}$$

Capacitor energy or the self-energy of electron charge.



$U = q/C$