Lecture 03

01/24/23

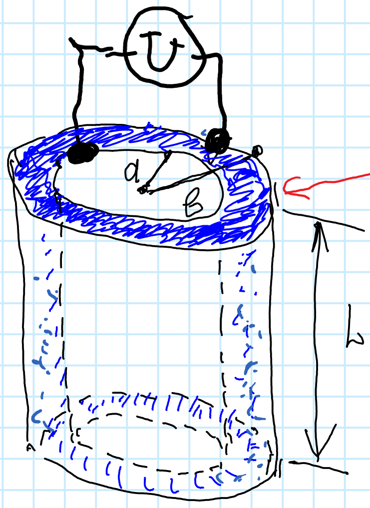
Examples of EM Problems and their Solutions

Ohm's Law: $\boxed{U = RI \text{ or } \vec{j} = \sigma \cdot \vec{E}}$ (σ – conductivity)

EM Induction Law: $\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \text{ or } \vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}}$

Problem 1: Ohm's law and resistance

Example: Calculate the resistance R of the segment of two coaxial cylinders. The voltage U is applied between a and b surfaces.



σ – conductivity

I – current ; U – voltage

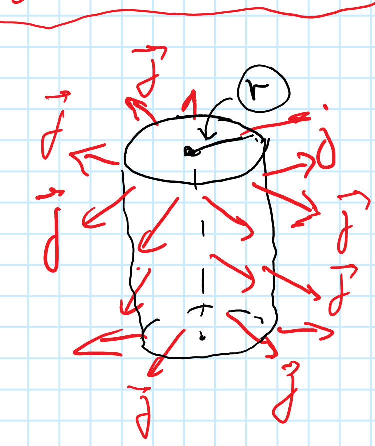
Ohm's Law: $\boxed{R = U/I}$

We can calculate the electric current I and difference of potentials $\varphi(r_a) - \varphi(r_b) = U$, if the electric charge is distributed on the surface of the inner cylinder of the radius "a". We can assign the linear charge density λ for the inner cylinder charge distribution.

recap: Gauss' Theorem $\oint_S \vec{E}\, d\vec{S} = E \cdot 2\pi r L = \frac{q}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0}$

Electric field for the region $a < r < b$: → $\boxed{E = \frac{\lambda}{2\pi\varepsilon_0 r}}$

and the density of the electric current $\vec{j}$ → $\boxed{j = \sigma E = \frac{\lambda\sigma}{2\pi\varepsilon_0 r}}$



The difference of potentials U:

$$U = \int_a^b E\,dr = \frac{\lambda\sigma}{2\pi\varepsilon_0}\int_a^b \frac{dr}{r} = \frac{\lambda\sigma}{2\pi\varepsilon_0}\ln b/a;$$

The electric current I: $I = j \cdot 2\pi r L = \frac{\lambda \cdot \sigma}{\cancel{2\pi\varepsilon_0 r}} \cdot \cancel{2\pi r} L = \lambda\sigma L/\varepsilon_0$

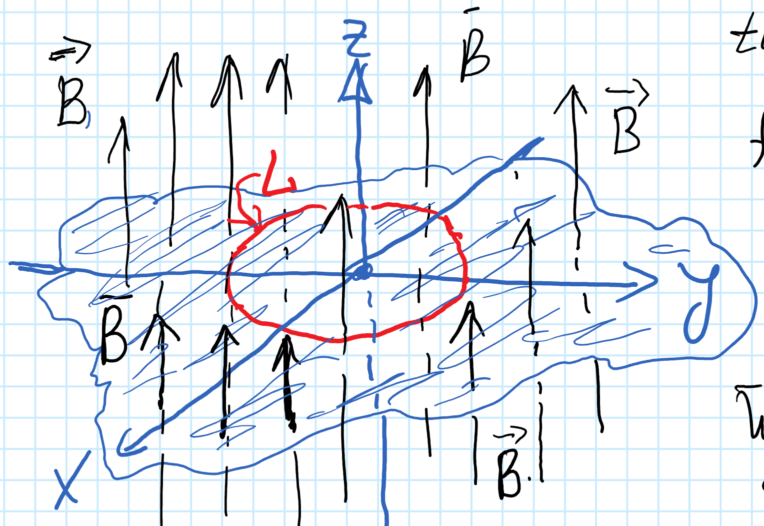
The resistance R: $R = \frac{U}{I} = \frac{\cancel{\lambda}\sigma \ln b/a \cdot \cancel{\varepsilon_0}}{2\pi\cancel{\varepsilon_0}\,\cancel{\lambda}\cdot\sigma L} \Rightarrow \boxed{R = \frac{\ln b/a}{2\pi\sigma L}}$

Problem 2:

Find the electric field $\vec{E}$ induced by the cylindrically symmetric magnetic field: $\boxed{\vec{B}(\vec{r},t) = \hat{e}_z \cdot B_0 \cdot \cos\omega t}$

the unit vector of the z-axis; time "t"

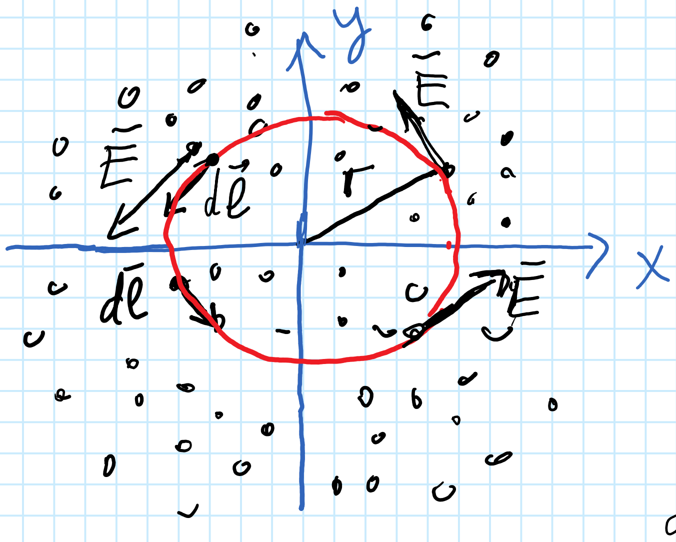
B_0, ω ≡ constants



We will discussed two methods of solution.

$\boxed{\text{Method 1}}$ Calculation of emf: $\oint_L \vec{E}\,d\vec{l} = -\frac{\partial}{\partial t}\underbrace{\int \vec{B}\,d\vec{S}}_{\Phi}$

The magnetic field $\vec{B}$ is uniform and we can use a circular loop "L" to calculate emf (the red circle in the figure)



$$\oint_L \vec{E}\,d\vec{l} = E(r)\cdot 2\pi r = -\frac{d}{dt}\int_S \vec{B}\,d\vec{s} = -\pi r^2 \frac{dB}{dt}$$

due to cylindrical symmetry

$\vec{B} \uparrow\uparrow \hat{e}_z$ and $d\vec{s} = ds\cdot\hat{e}_z$

$$E(r)\cdot 2\pi r = -\pi r^2 \frac{dB}{dt} \Rightarrow \boxed{E(r) = -\frac{r}{2}\frac{dB}{dt}}$$

$$\frac{dB}{dt} = \frac{d}{dt}(B_0 \cos\omega t) = -\omega B_0 \sin\omega t$$

$$\boxed{E(r) = \frac{\omega r B_0}{2}\sin\omega t}$$

Method 2: Direct calculation of the vector $\vec{E}$ from the equation describing the Faraday's Law of EM induction:

$$\boxed{\vec{E} = -\vec{\nabla}\varphi - \frac{\partial\vec{A}}{\partial t}}$$

$\varphi = 0$
$$\boxed{\vec{E} = -\frac{\partial\vec{A}}{\partial t}}$$

$$\vec{A} = \frac{1}{2}(\vec{B}\times\vec{r}) = \frac{1}{2}Br\cdot\hat{e}_\phi$$

$$\vec{E} = +\frac{1}{2}r\frac{\partial B}{\partial t}\cdot\hat{e}_\phi \Rightarrow$$

$$\boxed{\vec{E} = +\frac{\omega r B_0}{2}\sin\omega t\cdot\hat{e}_\phi}$$

In our Problem $\varphi = 0$. The vector potential of the cylindrically symmetric uniform magnetic field can be written as

$\vec{A} = \frac{1}{2}(\vec{B}\times\vec{r})$. Let us check it:

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{2}\vec{\nabla}\times(\vec{B}\times\vec{r}) = \frac{1}{2}\left[\vec{B}(\vec{\nabla}\cdot\vec{r}) - (\vec{B}\cdot\vec{\nabla})\vec{r}\right]$$

common vector algebra

$$\vec{a}\times\vec{b}\times\vec{c} = \vec{b}(\vec{a}\vec{c}) - \vec{c}(\vec{a}\vec{b}) = \vec{b}(\vec{a}\vec{c}) - (\vec{b}\vec{a})\vec{c}$$

$\vec{B}$, $\vec{\nabla}$, $\vec{r}$, $\vec{B}$, $\vec{\nabla}$, $\vec{r}$

$$\vec{\nabla}\cdot\vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3; \qquad (\vec{B}\cdot\vec{\nabla})\vec{r} = B\frac{d}{dz}\vec{r} = \vec{B}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{2}(3\vec{B} - \vec{B}) = \vec{B} \leftarrow \text{All is correct!}$$

For the cylindrical coordinates (r, ϕ, z):

$$\vec{A} = \frac{1}{2}(\vec{B}\times\vec{r}) = \frac{1}{2}Br\cdot\hat{e}_\phi$$