Physics 1601 Take-home Mid-term Exam, Spring 2014
PLEASE JUSTIFY/COMMENT ON ALL OF YOUR ANSWERS
AND PRESENT YOUR SOLUTIONS CLEARLY!

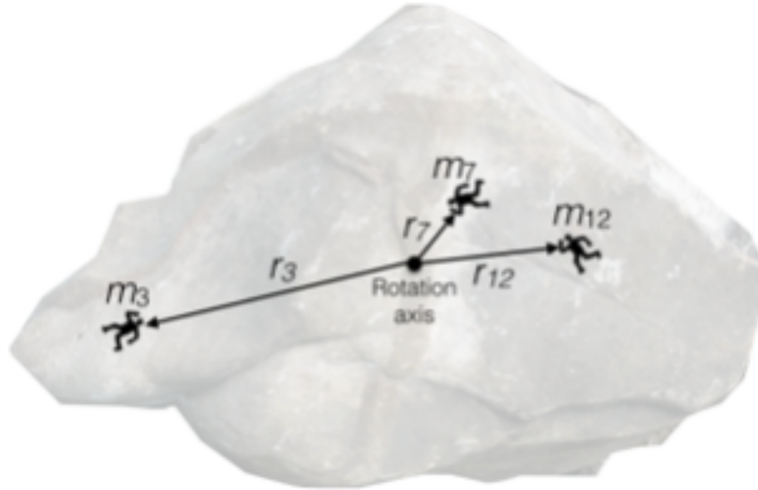
Midterm #2 Solutions

Physics 1601, Spring 2014

There are often many ways to arrive at an answer. For some conceptual problems, it happens through the intuition we have gained since birth about how the natural world works. For problems which use mathematics, there are often multiple approaches that works equally well. This solution set is only one possible account of reasoning through the test.

Inner monolog in italics

1. Katniss Everdeen from district 12 (mass m_{12}=60 kgs) and two of her allies, from districts 3 (m_3=75 kg) and 7 (m_7=90 kg), are standing on a rock at rest in the center of a lake. The tributes brace themselves by lying down on the rock to maximize their frictional force which prevent them from falling in the lake. The rock surface is uniform, but the tributes are dressed differently, so the coefficient of friction between the tribute's bodies and the rocks are μ_{12}=0.15, μ_3=0.1, and μ_7=0.12 (subscripts denote districts). When the rock begins spinning, the three tributes are r_{12} = 10 feet, r_3=20 feet, and r_7=4 feet from the center.



(a) At time t=0, a constant torque of 2000 Nm is applied to the rock's rotation axis. Assuming the moment of inertia of the rock is 5000 kg m², and that the tributes are very small and have negligible moments of inertia themselves, write a numerical expression for the angle $\theta(t)$ that the rock has rotated as a function of time.

The moment of inertia of the rock plus three tributes is:

$$I_{total} = I_{rock} + m_3 r_3^2 + m_7 r_7^2 + m_{12} r_{12}^2$$

The total angular acceleration is then determined from:

$$\tau = I_{total}\alpha_0 \qquad \omega(t) = \alpha_0 t \qquad \theta(t) = \frac{1}{2}\alpha_0 t^2$$

(b) What is the magnitude of the tangential linear acceleration of the tribute from district 3 ~~in a direction~~?

$$a_{tan} = r_3\alpha_0 = r_3\frac{\tau}{I_{total}}$$

(c) What is the magnitude of radial (aka centripetal) acceleration of the tribute from district 3?

$$a_{rad} = \frac{v_3(t)^2}{r_3} = r_3\omega(t)^2 = r_3(\alpha_0 t)^2$$

(d) Combine the two components of acceleration in (b) and (c) to find the total acceleration of the tribute from district 3 as a function of time.

$$a_{total} = \sqrt{a_{tan}^2 + a_{rad}^2} = r_3\sqrt{(\alpha_0 t)^4 + \alpha_0^2}$$

(e) At what time does the tribute from district 3 begin to slip?

The total acceleration times mass is the total applied force. The total force on the tribute is due to the static frictional force

$$F_s \leq \mu_3 N_3 = \mu_3 m_3 g$$

At time t_3, the equality holds and is equal to the acceleration times mass

$$\mu_3 m_3 g = m_3 r_3 \sqrt{(\alpha_0 t_3)^4 + \alpha_0^2}$$

$$t_3 = \alpha_0^{-1}((\frac{\mu_3 g}{r_3})^2 - \alpha_0^2)^{\frac{1}{4}}$$

(f) Find the angular acceleration of the rock after the last tribute has slipped and fallen into the lake.

$$\alpha_3 = \frac{\tau}{I_{rock}}$$

(g) Who slips first: Katniss or the tribute from district 7?

One way is to compute the times the two tributes slip at a given angular acceleration.
In more detail than is necessary:

The tribute from district 3 was the first to slip into the lake because he/she had the smallest coefficient of friction and was furthest from the center of rotation. The angular acceleration was then

$$\alpha_1 = \frac{\tau}{I_{rock} + m_7 r_7^2 + m_{12} r_{12}^2}$$

If Katniss slipped first, she would slip at a time

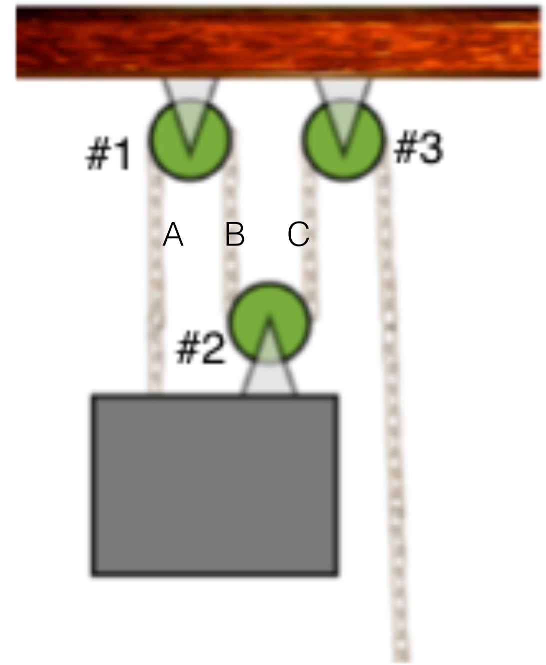
$$t_{12} = \alpha_1^{-1}\left(\left(\frac{\mu_{12} g}{r_{12}}\right)^2 - \alpha_1^2\right)^{\frac{1}{4}}$$

If the district 7 tribute slipped first, he/she would slip at a time

$$t_7 = \alpha_1^{-1}\left(\left(\frac{\mu_7 g}{r_7}\right)^2 - \alpha_1^2\right)^{\frac{1}{4}}$$

Katniss slips first

2. *Analytical:* A massless chain is strung through a system of ***massive*** pulleys (mass m_p, radius r) which suspends a mass M as shown. (*Hint: see the solutions to the massless pulley case from the first midterm on the syllabus website*)

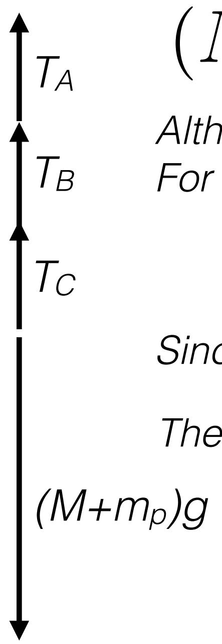


(a) What length of chain must be pulled to lift the mass by a height h?

To lift the mass a height h, the three segments of chain labeled A, B, and C must each get shorter by a distance h. Since the length of the chain doe not change, a chain length 3h must be pulled downward from the fourth segment

(b) What is the tension in the chain if the chain is held fixed?

The massive pulley is attached to the large mass M. The free body diagram for the hanging mass plus pulley implies that



$$(M + m_p)a_y = T_A + T_B + T_C - (M + m_p)g$$

Although the pulleys are massive, they do not rotate, meaning that the net torque about each one is zero. For the static case, then we have

$$T_A = T_B = T_C = T$$

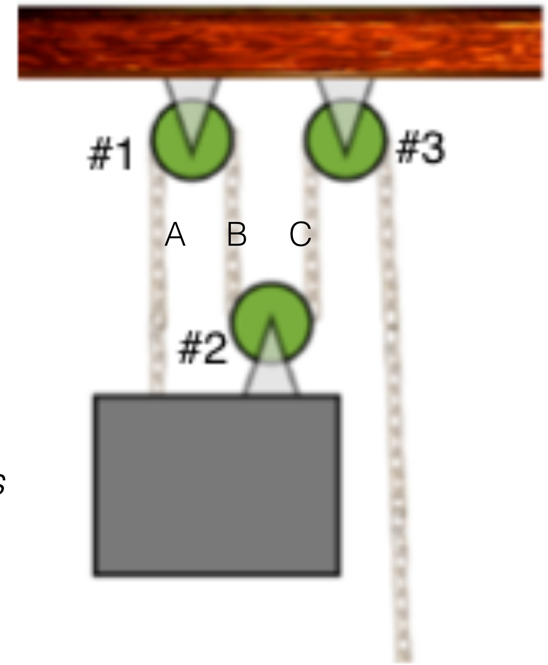
Since this part is a statics problem, there is no acceleration and a_y=0.

Therefore,

$$T = \frac{(M + m_p)g}{3}$$

In words, the weight of the hanging mass and attached pulley is shared equally among the chain links that support it.

(c) Another mass *m* is attached to the chain and is let go. How are the angular speeds of the three pulleys (ω_1, ω_2, ω_3) related?



If the mass drops a height H, and the pulley has a radius r, a length H of chain must move over the pulley #1 as it unwinds by an angle:

$$\theta_1 = \frac{H}{r}$$

The length of rope that rolls under pulley number two must be enough that the segments A and B shorten by a length H

$$\theta_2 = \frac{2H}{r}$$

Similarly, the length of chain that moves over the pulley #3 must be enough to shorten lengths A, B, and C, so

$$\theta_3 = \frac{3H}{r}$$

Taking one time derivative, the angular speeds are related:

$$\omega_3 = 3\omega_1 \qquad \omega_2 = 2\omega_1$$

(d) After release, the mass *m* moves vertically upward. Find the speed of the larger mass after it has fallen a height *H*.

Relating speeds and fallen heights? Energy conservation is the easiest approach:

$$E_0 = (M + m_p)gH$$

$$E_f = \frac{1}{2}mv_m^2 + \frac{1}{2}(M + m_p)v_M^2 + \frac{1}{2}I_{disk}(\omega_1^2 + \omega_2^2 + \omega_3^2) + mg3H$$

The angular speeds are related from the previous problem, and the speeds are related according to

$$v_m = 3v_M$$

$$E_f = \frac{1}{2}m(3v_M)^2 + \frac{1}{2}(M + m_p)v_M^2 + \frac{1}{2}(\frac{1}{2}m_p\cancel{r^2})((\frac{v_M}{\cancel{r}})^2 + (2\frac{v_M}{\cancel{r}})^2 + (3\frac{v_M}{\cancel{r}})^2) + mg3H$$

Collecting like terms and solving, we get:

$$v_M^2 = \frac{(M + m_p - 3m)gH}{\frac{9}{2}m + \frac{1}{2}M + 4m_p}$$

3. *Numerical* A mechanical stopwatch consists of a uniform disk ($I_{cm}=1/2MR^2$) of radius R=3cm and mass M=100g with a single second hand of mass m=20g and or length L=2cm ($I_{cm}=1/12ML^2$). The second hand rotates about an axis going through one end. The watch floats in space and is initially at rest.

(a) What is the moment of inertia of the second hand about its axis of rotation?

Using the parallel axis theorem, the MOI of the hand about the end is the MOI about the center plus m (L/2)^2

$$I_{end} = I_{cm} + m(\frac{L}{2})^2 = \frac{1}{3}mL^2$$

(b) The watch, initially at rest in space, starts ticking and the hand starts rotating clockwise. What is the total angular momentum of the watch?

The watch began at rest, so the angular momentum was zero. Then it began ticking, and parts rotating, but because there is no external torque, the angular momentum remains zero.

(c) Using the coordinate system shown in the figure, which direction does the angular velocity of the second hand point?

Using the right hand rule, and the fact that the hand rotates clockwise, the angular velocity is in the negative z direction.

(d) In the reference frame of the watch one revolution takes 60 seconds. The watch however moves as well to conserve angular momentum. How much time do you have to wait before the entire watch rotates through one full turn?

The total angular momentum is zero. In the reference frame of the stars, this can be expressed

$$L_{total} = 0 = I_h \omega_h - I_w \omega_w$$

The angular speed of the hand in the reference frame of the watch corresponds to once per 60 seconds, or

$$\omega_{h,w} = \frac{2\pi}{60 sec}$$

However, the time required to go one full revolution in the reference frame of the stars is longer, and so the angular speed is smaller by

$$\omega_h = \omega_{h,w} - \omega_w$$

$$I_h(\omega_{h,w} - \omega_w) = I_w \omega_w$$

$$\omega_w = \frac{I_h}{I_h + I_w} \omega_{h,w}$$

The period of the watch's rotation is then:

$$T_{watch} = \frac{2\pi(\frac{1}{3}mL^2 + \frac{1}{2}MR^2)}{\frac{1}{3}mL^2 \frac{2\pi}{60 sec}}$$