Physics 1601 Take-home Mid-term Exam, Spring 2014
PLEASE JUSTIFY/COMMENT ON ALL OF YOUR ANSWERS
AND PRESENT YOUR SOLUTIONS CLEARLY!

Midterm #1 Solutions

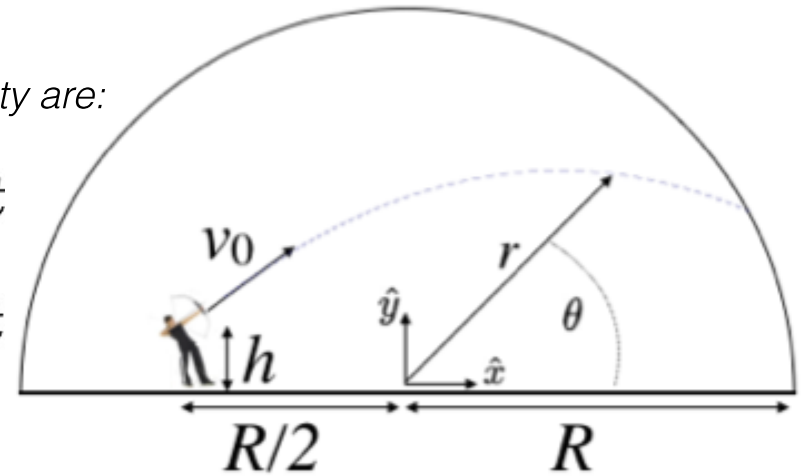
Physics 1501, Spring 2014

There are often many ways to arrive at an answer. For some conceptual problems, it happens through the intuition we have gained since birth about how the natural world works. For problems which use mathematics, there are often multiple approaches that works equally well. This solution set is only one possible account of reasoning through the test.

Inner monolog in italics

1. *Analyticial and Numeric*: Katniss Everdeen shoots an arrow that destroys the dome (radius *R*) she and others are trapped in. Her arrow leaves the bow with a speed v_0 which makes an angle of 60 degrees with respect to the ground from her shoulder height *h*, a distance *R*/2 from the center of the dome, as shown (figure not to scale).

(a) Write equations for the cartesian coordinates (*x*,*y*) of Katniss's arrow as functions of time assuming the center of the dome is the coordinate origin.



This is a 2D kinematics problem with constant acceleration.
The relevant equations for the x and y components of position and velocity are:

$$x(t) = x_0 + v_{0,x}t + \frac{1}{2}a_x t^2 \qquad v_x(t) = v_{0,x} + a_x t$$

$$y(t) = y_0 + v_{0,y}t + \frac{1}{2}a_y t^2 \qquad v_y(t) = v_{0,y} + a_y t$$

For the coordinate system given, gravity acts in the negative y direction, and not at all in the x direction, so

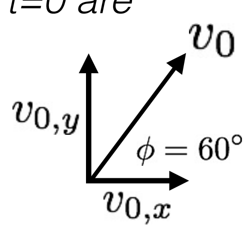
$$a_x = 0 \qquad a_y = -g$$

The initial position components at time t=0 are

$$x(0) = -\frac{R}{2} = x_0 \qquad y(0) = h = y_0$$

The initial velocity components at time t=0 are

$$v_x(0) = v_{0,x} = v_0 \cos\phi$$
$$v_y(0) = v_{0,y} = v_0 \sin\phi$$

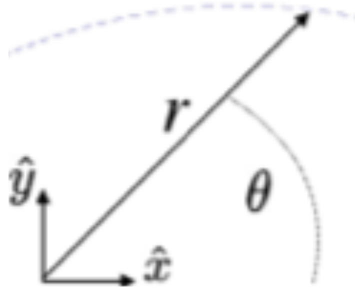


The equations for the trajectory, position versus time are:

$$x(t) = -\frac{R}{2} + v_0 t \cos\phi$$

$$y(t) = h + v_0 t \sin\phi - \frac{1}{2}gt^2$$

(b) Transform the coordinates into polar form (r,θ), according to the convention that the angle is zero when the arrow crosses the x axis and increases counterclockwise (as shown). Using symbols only, write and then sketch the independent expressions for r and θ as functions of time.



Polar coordinate transformations are similar to the vector components we have done so far.

$$x = r\cos\theta \qquad y = r\sin\theta$$

Which imply

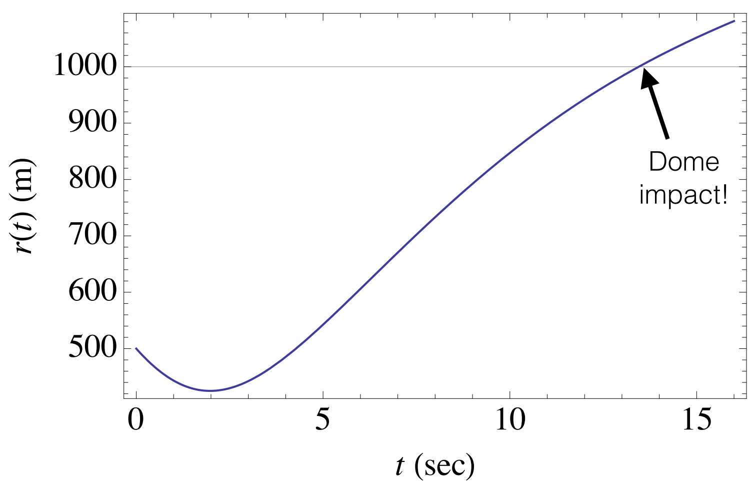
$$x^2 + y^2 = r^2 \qquad \theta = \arctan\frac{y}{x}$$

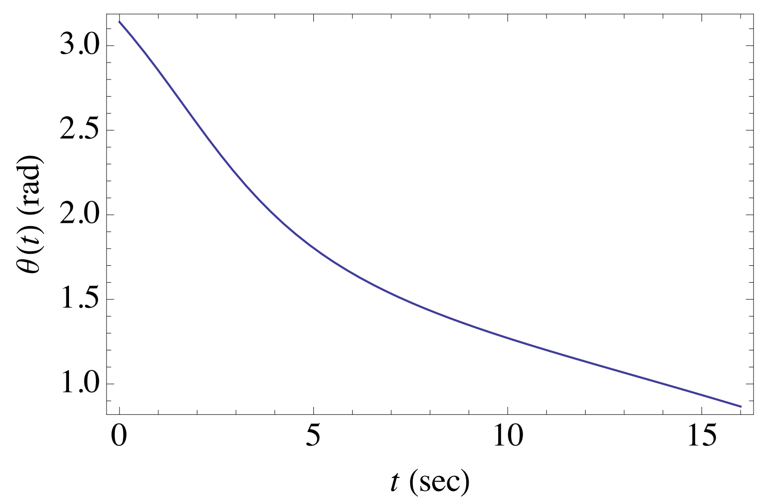
Putting our results in:

$$r(t) = \sqrt{(-\frac{R}{2} + v_0 t\cos\phi)^2 + (h + v_0 t\sin\phi - \frac{1}{2}gt^2)^2}$$

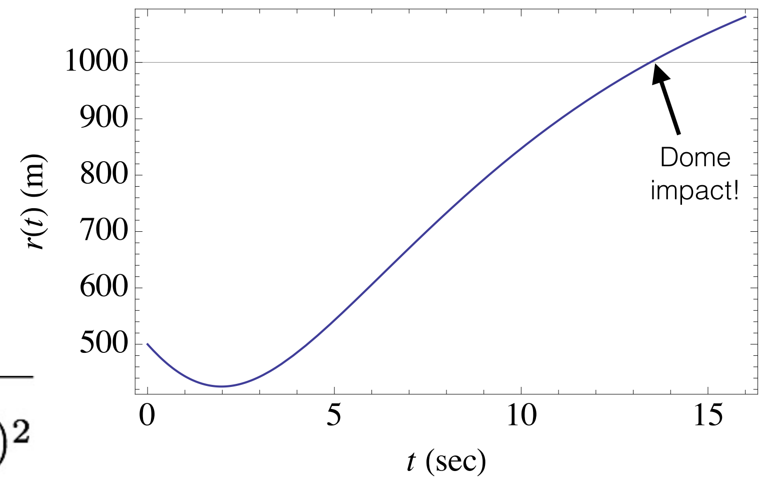
$$\theta(t) = \arctan\frac{h + v_0 t\sin\phi - \frac{1}{2}gt^2}{-\frac{R}{2} + v_0 t\cos\phi}$$

Plots for the values given





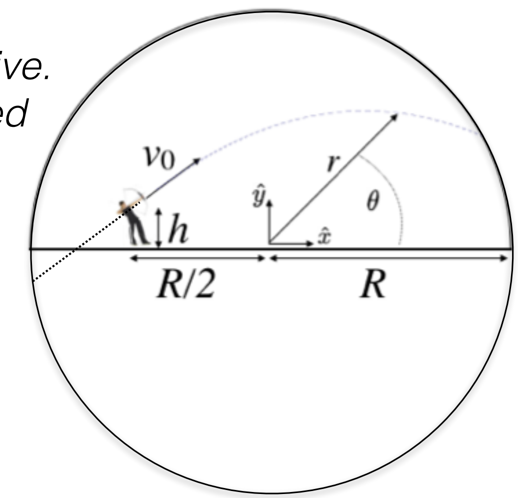
(c) Assuming R=1000m, v_0=150m/s, h=1m, find the angle θ and elevation y of the point where the arrow intercepts the dome.



One approach is to find the time of flight before impact (around 13.5 seconds in the graph) and plug it into the expressions for theta and y.

$$r(t_h) = R = \sqrt{(-\frac{R}{2} + v_0 t_h \cos\phi)^2 + (h + v_0 t_h \sin\phi - \frac{1}{2} g t_h^2)^2}$$

There are four numerical solutions. Two are unphysical and correspond to complex numbers for the times of impact. Of the other two, one is positive and the other is negative. The negative one corresponds to rewinding the trajectory to times before it was launched where it was a distance R from the origin (see figure).

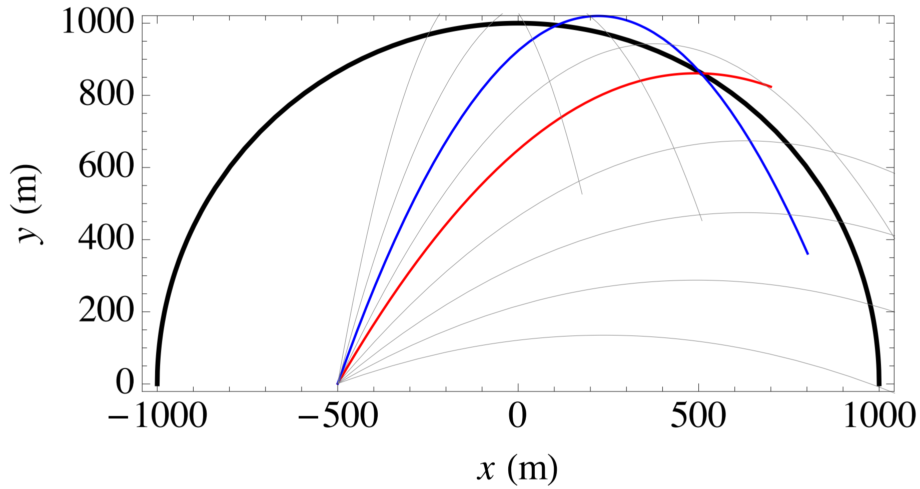


The positive solution is 13.4508 seconds. The corresponding values of theta and y are:

$$y(t_h) = 860.878m$$

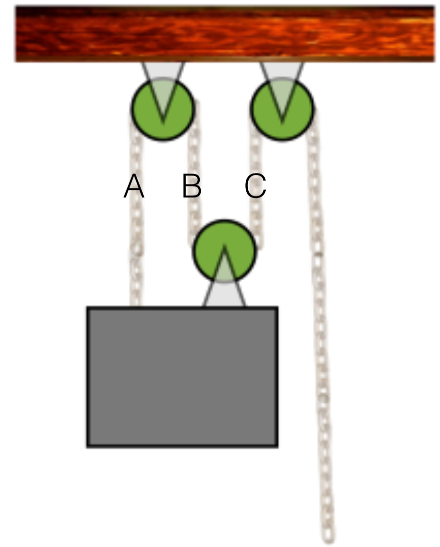
$$\theta(t_h) = 1.03699 rad = 59.4153°$$

(d) For the values given in part (c), is there another angle besides 60 degrees that Katniss could have chosen to shoot the arrow so that it would fly through the air and *directly* hit the same spot on the dome? If so, identify which angle would have the longer time of flight before hitting its mark.



The easiest approach to this is to just graph the trajectory for different values of the launch angle and see what happens. In the above graph, the red trajectory is the one that Katniss chose. There is another trajectory which intercepts the same point (the blue one), but it hits the dome in a different place first. The answer is that there is no second trajectory with the same initial speed that directly hits the dome in the same spot.

2. *Analytical:* A massless chain is strung through a system of massless pulleys which suspends a mass M as shown.

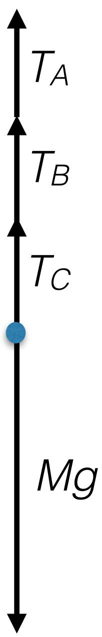


(a) What length of chain must be pulled to lift the mass by a height h?

To lift the mass a height h, the three segments of chain labeled A, B, and C must each get shorter by a distance h. Since the length of the chain does not change, a chain length 3h must be pulled downward from the fourth segment

(b) What is the tension in the chain if the chain is held fixed?

The free body diagram for the hanging mass implies that



$$Ma_y = T_A + T_B + T_C - Mg$$

Because the pulleys are massless, even when the hanging mass is moving, the tensions are equal:

$$T_A = T_B = T_C = T$$

Since this part is a statics problem, there is no acceleration and a_y=0.

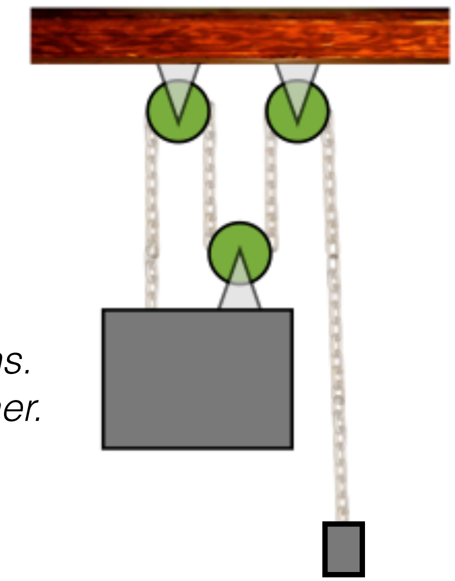
Therefore,

$$T = \frac{Mg}{3}$$

In words, the weight of the hanging mass is shared equally among the chain links that support it.

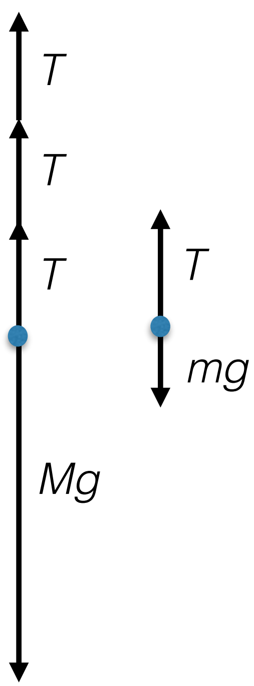
(c) **By pulling the chain, how much work must be done to lift the mass by a height *h*?**

Since there are no nonconservative forces acting, the work required to lift the mass is the change in potential energy: Mgh



(d) **Another mass *m* is attached to the chain and is let go. The mass *m* moves vertically upward. Find the acceleration of each mass.**

The problem is a kind of advanced Atwood's machine, similar to some homework problems. Because the pulleys are massless, the tension in the segments are still equal to one another. This would change if they were massive, because then there would have to be an unbalanced torque to rotate the pulleys. For now,



$$Ma_{M,y} = 3T - Mg \qquad \text{(i)}$$

$$ma_{m,y} = T - mg \qquad \text{(ii)}$$

From part (a), movement of the mass M downward implies movement of m upward by 3 times the distance. The speed and acceleration have the same relation, so:

$$a_{m,y} = -3a_{M,y} \qquad \text{(iii)}$$

Eliminate tension by combining equations (i)-3x(ii):

$$Ma_{M,y} - 3ma_{m,y} = -Mg + 3mg$$

Use (iii)

$$a_{m,y} = 3g\frac{M - 3m}{9m + M} \qquad a_{M,y} = -g\frac{M - 3m}{9m + M}$$

(e) Find the tension in the chain while the mass falls.

Solving for the tension from the same equations gives

$$T = Mg\frac{4m}{M+9m}$$

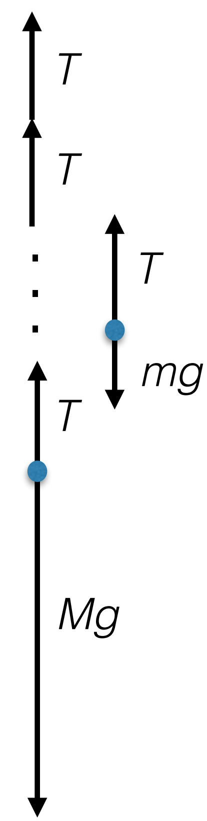
(f) Generalize you answers in (a), (b), (c), and (d) to a case where there are *N* such pulleys, assuming *N* is odd and (*N*-1)/2 are attached to the block and (*N*+1)/2 are attached to the ceiling (*N*=3 in the picture).

(a) Nh

(b) Mg/N

(c) Mgh

(d) The free-body diagram and equations are nearly the same:



$$Ma_{M,y} = NT - Mg$$

$$ma_{m,y} = T - mg$$

$$a_{m,y} = -Na_{M,y}$$

The solutions are:

$$a_{m,y} = Ng\frac{M-Nm}{M+N^2m}$$

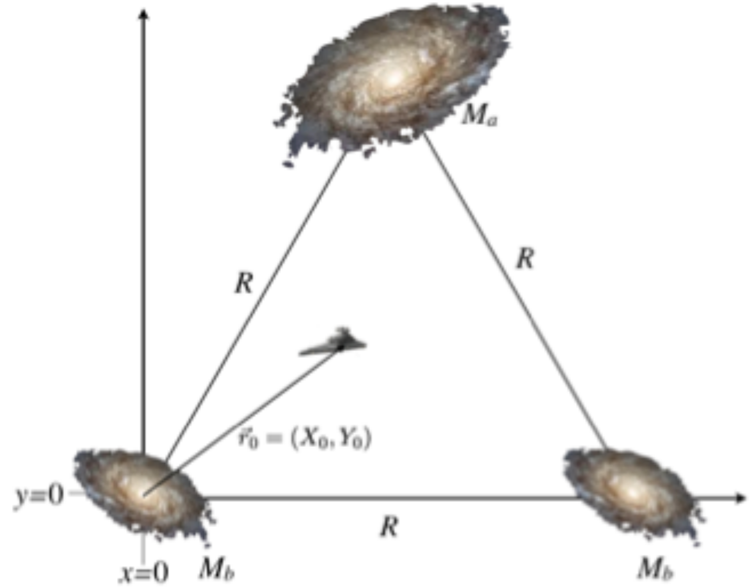
$$a_{M,y} = -g\frac{M-Nm}{M+N^2m}$$

$$T = \frac{Mmg(N+1)}{M+N^2m}$$

BTW, notice that by adding pulleys, a heavy mass was lifted with a fraction of the force required to lift it directly. This is the principle of a compound bow and other lifting devices. However, there is no free lunch. The work required to lift the mass is still *Mgh.* So while the force required is reduced to *Mg*/*N*, that force must act over a longer length of chain *Nh*. The total work is then (*Mg*/*N*)x(*Nh*)=*Mgh*

3. *Analytical:* Three galaxies reside on the corners of an equilateral triangle with side *R*. The two galaxies at the base of the triangle have mass M_b, and the galaxy at the apex has a mass M_a.

(a) Find the point X_0, Y_0 where the net gravitational pull on a spaceship is zero. Check that it is reasonable in any limits you can think of.

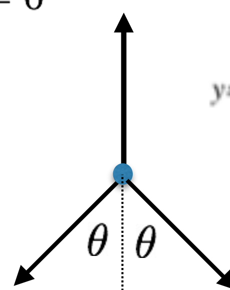


The x component of the zero-gravity point is $X_0=R/2$ by symmetry: It has be along the high-symmetry line because why would it be one side or the other, when the masses are symmetrically placed? Given this, the y component is the solution to

$$F_{NET,y} = -\frac{GM_bm}{X_0^2+Y_0^2}\cos\theta - \frac{GM_bm}{X_0^2+Y_0^2}\cos\theta + \frac{GM_am}{(\frac{\sqrt{3}}{2}R-Y_0)^2} = 0$$

$$0 = -2\frac{M_b}{X_0^2+Y_0^2}\frac{\frac{R}{2}}{\sqrt{X_0^2+Y_0^2}} + \frac{M_a}{(\frac{\sqrt{3}}{2}R-Y_0)^2}$$

$$R\frac{M_b}{M_a}(\frac{\sqrt{3}}{2}R-Y_0)^2 = ((\frac{R}{2})^2+Y_0^2)^{\frac{3}{2}}$$



(b) Write an expression for the net gravitational force vector on a mass *m* at a general point (*x*,*y*).

The force is the sum of gravitational forces from the three galaxies. These contributions come from Newton's universal gravitation, but the sources are at slightly different positions and the directions of each need to be resolved. On way of writing these is:

$$F_{b,left} = -\frac{GM_bm}{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2}}\hat{j}\right)$$

$$F_{b,right} = -\frac{GM_bm}{(x-R)^2+y^2}\left(\frac{x-R}{\sqrt{(x-R)^2+y^2}}\hat{i} + \frac{y}{\sqrt{(x-R)^2+y^2}}\hat{j}\right)$$

$$F_a = -\frac{GM_bm}{(x-R/2)^2+(y-\sqrt{3}R/2)^2}\left(\frac{x-R/2}{\sqrt{(x-R/2)^2+(y-\sqrt{3}R/2)^2}}\hat{i} + \frac{y}{\sqrt{(x-R/2)^2+(y-\sqrt{3}R/2)^2}}\hat{j}\right)$$

(c) Consider the limit where the spaceship is outside the triangle and far away. Specifically, determine an expression for the strength of the gravitation force of the three galaxy cluster when the spaceship is a distance $y \gg R$ along the y axis and directly above the galaxy at the apex.

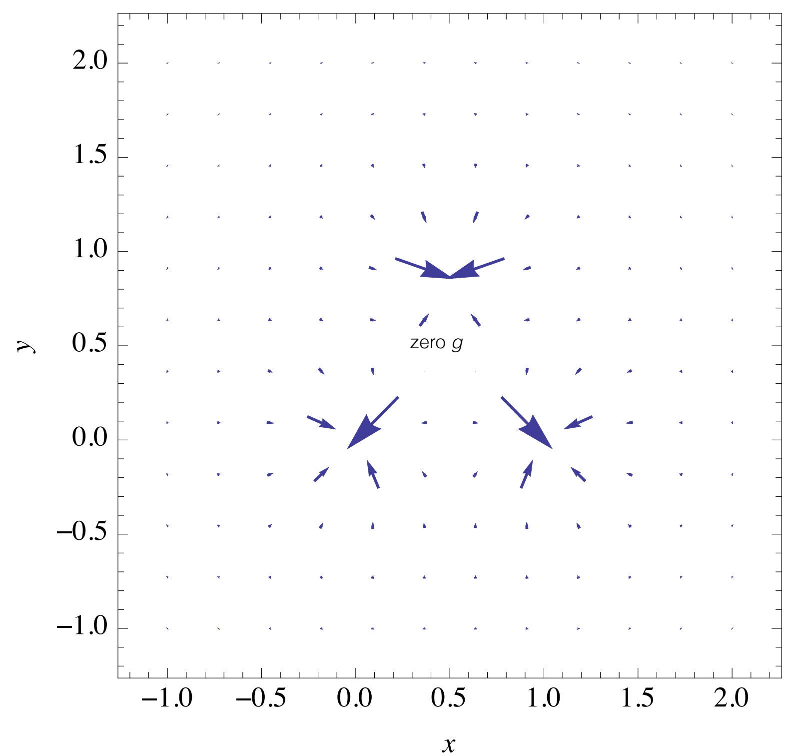
If the space ship is far away, the net pull of the galaxies will be close to the pull of one galaxy with the sum of the three masses.

$$F_{faraway} = -\frac{G(M_a + 2M_b)m}{y^2}\hat{j}$$

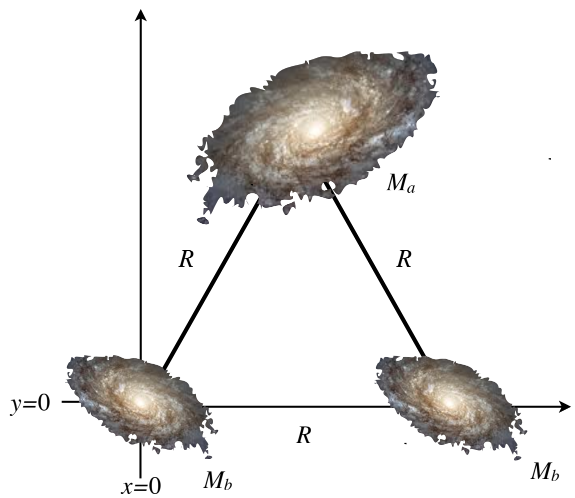
BTW, This is the "lowest order" effect. If additional precision is needed, one can "expand" the functions in the previous example. This expansion term is called the "monopole", the next order term is the "dipole", then "quadrupole", "octopole", "hexadecapole", etc. These higher order terms, which are due to the detailed arrangement of the masses, or charges in the case of electromagnetism, are described in terms of higher order "tensors", and are widely used in physics

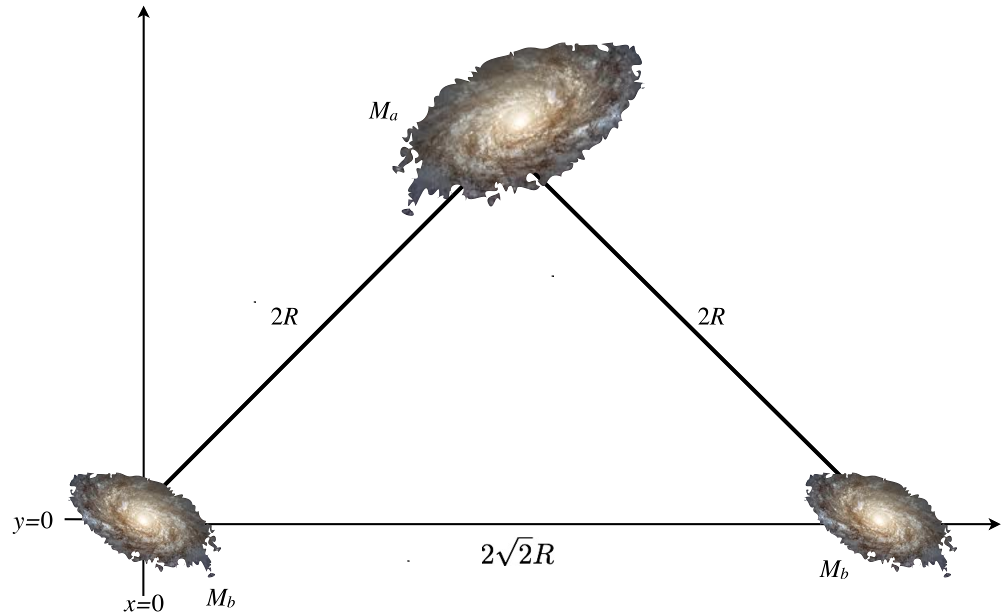
(d) Assuming that $M_a = 2M_b$, sketch by hand, or using a plotting program, the gravitational field. That is, draw at representative points on a graph the direction and magnitude of the net force on a test mass that is placed at that point. Try to be as accurate as possible and be sure to include points outside and inside the triangle near and away from each galaxy as well as labeling clearly, and to scale, the point(s?) where gravity is zero.

Points outside the triangle point into it, and those near galaxies point toward the galaxy. The zero-g point is slightly above the center of the triangle.



(e) You have a dream that you are a sentient super-being being capable of manipulating galactic clusters by hand and you decide to rearrange these galaxies to reside on the three corners of a square with side $2R$. How much work did you do to accomplish this?





The work required to rearrange the galaxies is the change in potential energy. Initially, the potential energy is

$$U_0 = -\frac{GM_b^2}{R} - \frac{2GM_bM_a}{R}$$

Afterward, the PE is

$$U_f = -\frac{GM_b^2}{2\sqrt{2}R} - \frac{2GM_bM_a}{2R}$$

The work required to make the rearrangement is:

$$W_{\triangle\to\square} = \frac{GM_b^2}{R}\left(1 - \frac{1}{2\sqrt{2}}\right) - \frac{GM_bM_a}{R}$$

(f) Imagine a spaceship which is starts at the point $(10{,}000R, R/2)$, headed to the left ($-x$ direction). Is this information enough to uniquely determine the trajectory? If so, draw it. If not, draw a few possibilities. Explain your reasoning.

The speed is critical missing information. If the speed is very high, then its total energy may be greater than 0 and then the spaceship will be unbound from the galactic cluster, making just one pass. If the speed is sufficiently low, then the spaceship will orbit a few times, and possibly crash into a galaxy.

4. *Numeric:* The final stage of a space shuttle (M=2 million kg) launch sequence involves the shuttle's release of the empty external tank (m=350,000 kg). Initially, the shuttle and tank move at a constant speed v_0=7800m/s at an angle θ=10 degrees from the vertical. A small explosion pushes the tank away from the shuttle for safety and alters the trajectory of both.

Neglect the effects of gravity in parts (a), (b), and (c). You may need the earth's mass and radius, which you can find on your own.

(a) If the shuttle is now moving v_s=7700m/s vertically, what are is horizontal component of the tank's velocity v_t?

There are no external forces (neglect gravity), so both horizontal (x) and vertical (y) components of momentum are conserved.

$$p_{0,x} = (M+m)v_0 \sin\theta = mv_{t,x}$$

$$v_{t,x} = \frac{(M+m)v_0 \sin\theta}{m}$$

(b) What is the vertical component of the tank's velocity?

$$p_{0,y} = (M+m)v_0 \cos\theta = mv_{t,y} + Mv_s$$

$$v_{t,y} = \frac{(M+m)v_0 \cos\theta - Mv_s}{m}$$

(c) How much work was done by the explosion on the tank and shuttle to separate these components?

The work done by the explosion is the change in kinetic energy

$$W = K_f - K_i = \frac{1}{2}M(v_{t,x}^2 + v_{t,y}^2) + \frac{1}{2}mv_s^2 - \frac{1}{2}(M+m)v_0^2$$

(d) After detachment from the tank, the shuttle uses rocket propulsion to ultimately situate itself in a circular orbit, where the shuttle orbits the earth directly above Houston at all times of the day, with which it communicates regularly. Draw a free body diagram for a shuttle in such an orbit, indicating the direction of earth's center. Find the radius of this "geosynchronous" orbit.

The gravitational force provides the centripetal acceleration.

$$Ma_x = M\frac{v^2}{R_g} = \frac{GM_eM}{R_g^2}$$

The speed is also related to the geosynchronous radius Rg:

$$v = \omega R_g$$

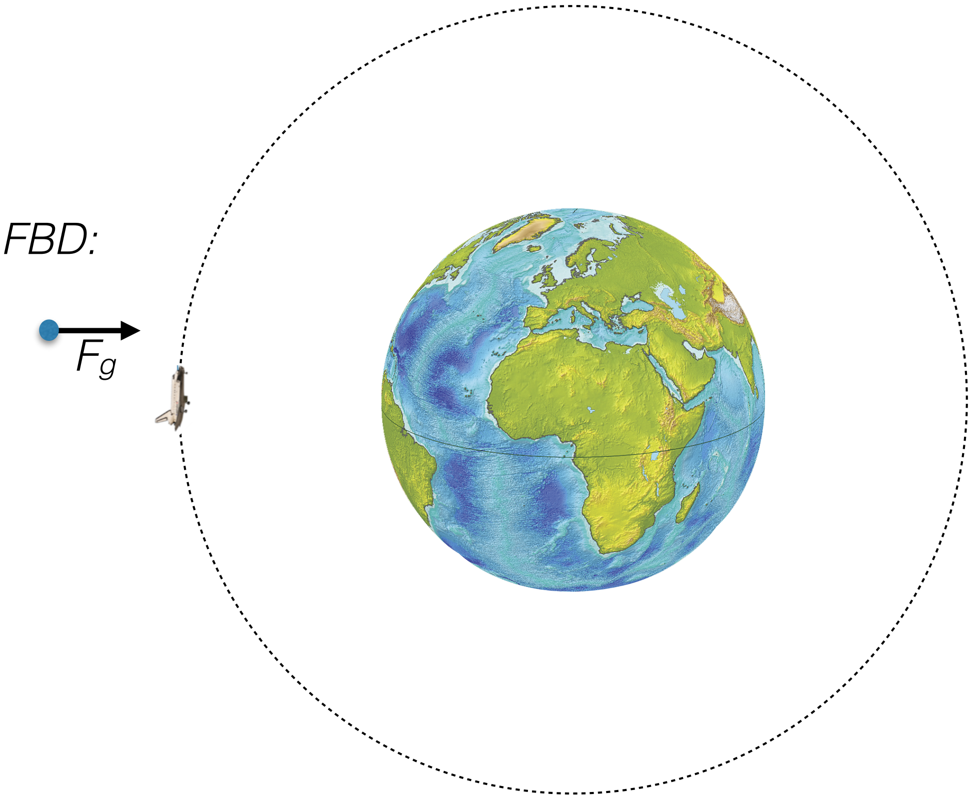
where

$$\omega = \frac{2\pi \text{ radians}}{1 \text{ day}}$$

Solving for Rg:

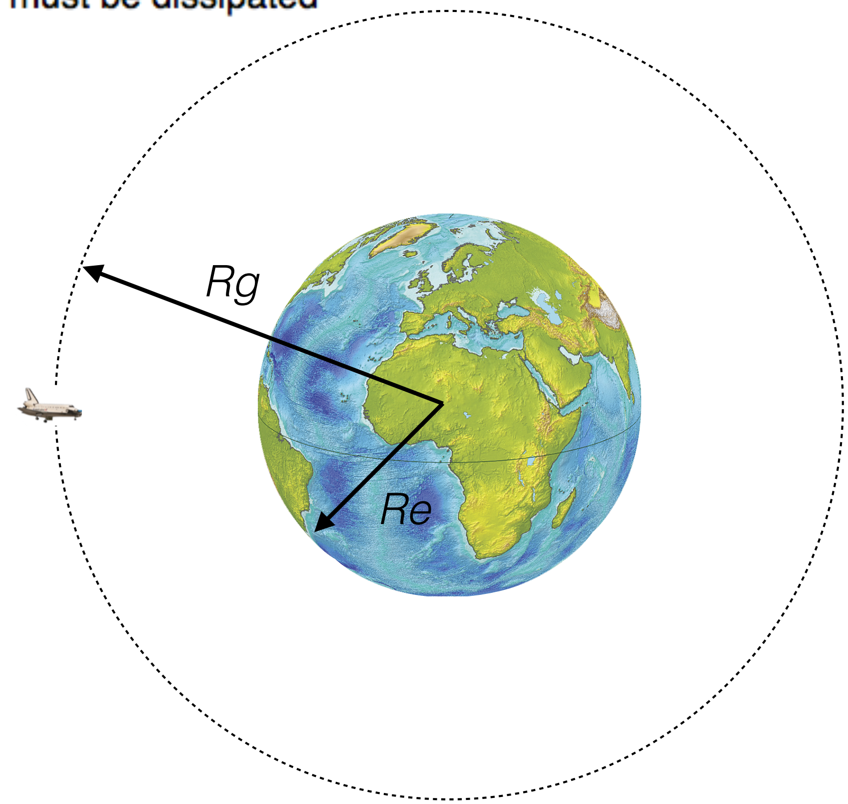
$$R_g = \left(\frac{GM_e}{\omega^2}\right)^{\frac{1}{3}}$$

FBD:



(e) The shuttle now re-enters the earth's orbit. Assuming for simplicity that the shuttle starts from rest at the geosynchronous orbit radius and falls directly into the ocean without any losses due to heat or other nonconservative influences, how fast would the shuttle hit the water? Convert your answer into a multiples of the speed of sound (aka "Mach" number), which is v_s=343 m/s. If the maximum speed that the shuttle could re-enter and the passengers could eject safely and survive is 300m/s, how much energy must be dissipated upon re-entry? Express your answer in gigajoules.

The speed of falling objects is best found through work-energy, since we neglect the nonconservative processes like heating due to atmosphere.



$$E_0 = -\frac{GMM_e}{R_g}$$

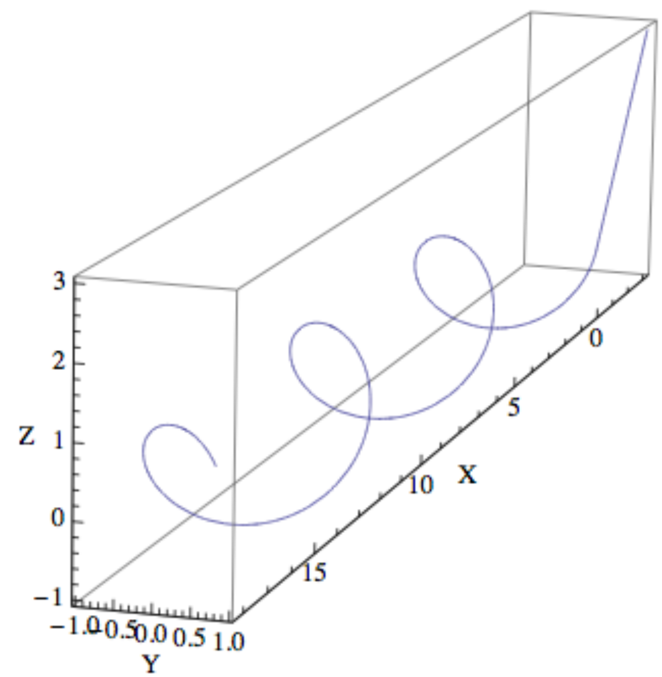
$$E_f = \frac{1}{2}Mv_f^2 - \frac{GMM_e}{R_e}$$

$$v_f = \sqrt{2GM_e(\frac{1}{R_e} - \frac{1}{R_g})}$$

If the maximum speed allowed for passengers to eject safely is v_e=300m/s, the energy required to be dissipated is

$$\frac{GMM_e}{R_e} - \frac{GMM_e}{R_g} - \frac{1}{2}Mv_e^2$$

5. *Essay*: The picture shows a segment of a roller coaster track with a particular shape, which twists and turns in all three dimensions (numerical values on the graph are not for use in the problem).



For X>0, the shape of the track has the form:
Y = R Cos(a X)
Z = - R Sin(a X)
which describes a helix.
The roller coaster cart is fed into this helical track segment by a ramp from X<0, with the equation
Y = R
Z = - a X.
The roller coaster slides along the frictionless track without slipping after being released from the point on the ramp (X<0) where Z=H.

Describe, using words, sketches, and graphs, the evolution of the ride if the cart is released from rest at a height $H<R$, $H=R$, and $H>R$. Explain your reasoning and carefully define your terms in your discussion.

Here are some points to consider for your discussion:
At what point(s) would a rider experience the greatest acceleration?
At what point(s) would the cart be going fastest?
At what point(s) would the cart be going slowest?
Describe in detail the behavior of the three components of the roller coaster cart's velocity and acceleration vectors at different points along the helix for each case.
Draw a graph of the "g-force" (aka acceleration) that riders would feel as a function of X.
If the radius R=10m, and the human body can safely withstand only 6g, what is the maximum height H that the cart can be released?