It's often convenient to use a scaled form of the Yen's by defining the spherical harmonic tensors:

\[ C^{(k)}_{\ell} \equiv \sqrt{\frac{4\pi}{2k+1}} Y_{\ell}^{k}(0,0) \]  

2) \text{Rank 0} -- just gives scalars & scalar operators

3) \text{Rank 1} --

Noting the form of the Yen's,

\[ Y_{10} \propto \frac{z}{r} \]

\[ Y_{1\pm 1} \propto \frac{(x \pm iy)}{\sqrt{2r}} , \]  

some proportionality constants.

It is clear that any vector operator can be expressed as a spherical tensor operator by taking its spherical components.

For a vector \( \mathbf{A} \),

\[ T^{(1)}_{0} = A_0 = A_z \]

\[ T^{(1)}_{\pm 1} = A_{\pm 1} = \mp \frac{1}{\sqrt{2}} (A_x \pm iA_y) \]

⇒ In particular, \( \mathbf{j} \) forms a rank 1 tensor operator, since \( j_0 = j_z \) and \( j_{\pm 1} \) satisfy (23).

⇒ So does the electric dipole operator, \( \mathbf{D} = e \mathbf{r} \mathbf{E} \), which we can write as,

\[ \mathbf{D} = e \mathbf{r} \mathbf{C}^{(1)} \]  

using (23).

Since we usually take matrix elements of writing \( \mathbf{E} = E_{\omega} \hat{\mathbf{E}} \), the spherical components correspond nicely to the polarization of a plane wave propagating in the \( z \)-direction (\( \pm 1 \)) or in the \( x-y \) plane (0).
\[ E_0 = E_z \quad \text{linear polarization along } z \]
\[ E_+ = -\frac{1}{\sqrt{2}} (E_x + iE_y) \quad \text{circular polarization } \sigma_- \]
\[ \quad \text{(propagation along } z) \]
\[ E_- = \frac{1}{\sqrt{2}} (E_x - iE_y) \quad \text{circular polarization } \sigma_+ \]

4) Higher rank -- the most common is the quadrupole, expressed most often as a multiple of \( \hat{C}_2 \). Note that there are 5 components to a spherical tensor, not 9 (with redundancies) as for a 2nd rank Cartesian tensor.

**Scalar Product**

The scalar or dot product is defined according to:

\[ \hat{f}, \hat{g} = \hat{f}^{(k)} \cdot \hat{g}^{(k)} = \sum_{\ell} (-1)^{\ell} q^\ell T^{(k)}_\ell U^{(k)}_{-\ell} \quad (25) \]

For vectors, if \( \hat{f}^{(1)} = \tilde{A} \)

\[ \hat{g}^{(1)} = \tilde{B} \]

\[ A_{\pm 1} = \pm \frac{1}{\sqrt{2}} (A_x \pm iA_y), \quad \text{etc.} \]

\[ T^{(1)} \cdot U^{(1)} = \sum (-1)^q A_q B_{-q} = \sum A_q^* B_q \]

\[ = A_x B_x + A_y B_y + A_z B_z, \quad \text{as expected.} \]

Finally, note that the unit vectors in spherical components are still orthogonal,

\[ \hat{e}_p^* \cdot \hat{e}_q = \delta_{pq} \]
From the commutation relations $[L_y, Y_{lm}]$, it can be shown that,

$$<L m | Y_{lm} | L' m' > = (-1)^m \int Y_{l-m} Y_{l'm'} Y_{l0} \, d\Omega$$

$$= (-1)^m \frac{(2l+1)(2l'+1)(2m+1)}{4\pi} \begin{pmatrix} l & l' & k \\ -m & m' & q \end{pmatrix} \begin{pmatrix} l & l' & k \\ 0 & 0 & 0 \end{pmatrix}$$

Only this term depends on $m'$.

By definition 26, the same is true for any matrix element $<j', m' | T^{(k)} | j, m >$.

We separate out the $m$-independent part and give it a special name, with the following phase convention:

$$<\alpha j m | T^{(k)} | \alpha' j' m'> = (-1)^{j-m} \begin{pmatrix} j & k & j' \\ -m & q & n' \end{pmatrix} <\alpha j || T^{(k)} || \alpha' j'>$

Wigner–Eckart Theorem

"Reduced matrix element"

$\alpha, \alpha'$ represent additional quantum numbers, like $n, l, s$ for states $|nlsm >$.

Significance:

If we express an interaction in spherical tensor notation, we can immediately separate it into a coupling coefficient that has all of the dependence on orientation and angular symmetry, and a factor that contains the detailed physics of the interaction!
Wigner - Eckart Theorem & Spherical Tensors - Version 2

A more satisfying approach may be to consider the behavior of operators under finite rotations.

For the angular momentum functions $|J_m \rangle$, a rotation $\hat{R}$ can only give a linear combination of other $M$ values,

$$\hat{R}(\phi, \theta, \chi) |J_m \rangle = \sum_{m'} D^{J}_{m'm}(\phi, \theta, \chi) |J_{m'} \rangle$$

The coefficients form a $(2J+1)^2$ unitary matrix, the rotation matrix, which will be discussed in detail later. We can write it as,

$$D^{J}_{m'm} = \langle J_{m'} | \hat{R} | J_m \rangle .$$

Now, the key idea is that if an arbitrary operator behaves like the angular momentum functions under rotations, then we can use all of the theorems & apparatus of angular momentum formalism to evaluate its matrix elements. Thus we define a spherical tensor operator as a set of $2J+1$ functions $T_{\ell}^{(k)}(\theta, \phi)$ that transform like $J$ or $Y_{m}$,

$$\hat{R}_{\ell} T_{\ell}^{(k)} \hat{R}^{-1} = \sum_{\ell'} D^{J}_{\ell\ell'} D_{\ell'}^{(k)} T_{\ell}^{(k)}$$

For $k = 0$, just ordinary scalars or operators - must be invariant under rotation by defn.

For $k = 1$, vector operators in spherical basis set etc. $Y_{m}$'s, etc. ($Y_{m}$'s can be functions of operators, just like $x, y, z$.)
Anyhow, on to the W-E theorem. Follow Zare's proof: For angular momenta,

$$|j_1 j_2 j m> = (-1)^{j_2-j_1-m} \sum_{m_1, m_2} \sqrt{2j_1+1} \binom{j_1 j_2 j}{m_1 m_2 -m} |j_1 m_1> |j_2 m_2>$$

So for an irreducible spherical tensor operator, we can write,

$$|\alpha' k j' n m> = (-1)^{j'-k-m} \sum_{g, m'} \sqrt{2k+1} \binom{k j' n}{g m' -m} \tilde{T}^{(k)}_g |\alpha' j' m'>$$

**Unrelated Quantum #'s**

Multiply by $<\alpha jm|$

$$<\alpha jm|\alpha' k j' n m> = \sum_{g, m'} \binom{k j' n}{g m' -m} \tilde{T}^{(k)}_g |\alpha' j' m>$$

Multiply by $\binom{k j' n}{g m' -m}$ and sum over $n, m$ (see AM-6)

$$\sum_{n, m} \binom{k j' n}{g m' -m} <\alpha jm|\alpha' k j' n m> = (-1)^{j'-k-m} <\alpha jm|\tilde{T}^{(k)}_g|\alpha' j'>$$

Zero unless $j = n$, $m = m'$. If so, have a scalar product independent of $m$ (i.e., rot. invariant.) We will call it $<\alpha j||\tilde{T}^{(k)}||\alpha' j'>$.
Application of W-E theorem — AM-16

Restrictions on permanent electric/magnetic moments

Here we will write \( \hat{d} \) = dipole operator (rank \( k=1 \))
\( \hat{Q} \) = quadrupole operator (rank \( k=2 \))

If the moment is "permanent" for a state \( \left| (\alpha) jm \right> \),
\[
\langle jm | \hat{d}^{(1)} | jm \rangle \neq 0 \text{ for some component } \alpha.
\]
(likewise for \( \hat{Q} \))

But,
\[
\langle \alpha jm | \hat{d}^{(1)} | jm \rangle = (-1)^j m q \begin{pmatrix} j & i \cr -m & q \end{pmatrix} \langle \alpha j | (\hat{d}^{(2)} | \alpha j \rangle
\]

\[
= 0 \text{ unless } q = 0
\]
\[
= 0 \text{ if } j = 0
\]

\( \Rightarrow \) No permanent dipole moment unless \( j \geq \frac{1}{2} \)
and the moment must point along the quantization axis.

For \( \hat{Q} \), rank 2 \( \Rightarrow \) must have \( j \geq 1 \);
only one moment, e.g., value of \( \hat{Q}^{(2)} \).

If we also take into account parity
(electric: \((-1)^k\), magnetic: \((-1)^{k-1}\))
then for a state of definite parity \( \alpha \) any
momentum (i.e., no degeneracy except \( m \)),
can have

1) Only even electric multipoles, for \( j \geq 1 \) only
2) Only odd mag. multipoles, for \( j \geq \frac{1}{2} \) only.

So, proton has a mag. moment, deuterium
has an electric quad. moment (\( I=1 \)), etc.
Example — Electric Dipole Selection Rules

Using (24), we need to find

\[ M = \langle n L S J m_j | r \ C_q^{(n)} | n' L' J' m_{j'} \rangle \ E_0 \]

From (27),

\[ M = E_0 (-1)^{J - m_j} \left( \begin{array}{c} J - 1 \ J' - 1 \\ m_j \ q \ m_{j'} \end{array} \right) \langle n L S J || r \ C_q^{(n)} || n' L' S J' \rangle \]

\[ \Delta(J 1 J') \ implies \ \left\{ \begin{array}{l} J' = J \pm 1 \ or \ J \pm 2 \\ \text{and} \ J + J' \geq 1 \\ \Sigma n' \ g' = 0 \end{array} \right. \]

\[ \Sigma n' = 0 \]

\[ \Sigma m' = 0 \]

\[ \Sigma q = -1 \]

\[ \Sigma q = 0 \]

\[ \Sigma q = 1 \]

\[ E_0 \ (-1)^{J - m_j} \left( \begin{array}{c} J - 1 \ J' - 1 \\ m_j \ q \ m_{j'} \end{array} \right) \langle n L S J || r \ C_q^{(n)} || n' L' S J' \rangle \]

Example — Landé Formula

Let \[ T^{(1)} = A \] (arbitrary vector operator)

\[ U^{(1)} = J \] (angular momentum op.)

By W-E Theorem,

\[ \langle \alpha j m | A q^{(n)} | \alpha j m' \rangle = \frac{\langle \alpha j || A || \alpha j \rangle}{\langle \alpha j || \hat{J} || \alpha j \rangle} = C \quad \text{ifd} \quad m' = \frac{m}{q} \]

"Evaluate" \( C \) by a trick — —

\[ \langle \alpha j m | A \hat{J} | \alpha j m' \rangle = \sum_{n'} \langle \alpha j m | A | \alpha j m' \rangle \langle \alpha j m' | \hat{J} | \alpha j m \rangle \]

\[ = C \sum_{n'} \langle \alpha j m | \hat{J} | \alpha j m' \rangle \langle \alpha j m' | \hat{J} | \alpha j m \rangle \]

\[ = C \langle \alpha j m | \hat{J}^2 | \alpha j m \rangle \]

\[ = C j (j+1) k^z \]

Thus, \[ \langle \alpha j m | A \hat{J} | \alpha j m' \rangle = k^z j (j+1) \]

Landé Formula
Finding reduced matrix elements

Since they don't depend on \( m \), just take a special case like \( m = m' = q = 0 \), evaluate \( \langle a_j m | T_q^{(k)} | a_j m' \rangle \), and divide by \((-1)^{j-m} \frac{j}{(j-m)} \). (a)

\[ \langle L' || Y_k || L \rangle \]

Taking \( \langle Lm | Y_{kq} | L'm' \rangle \) from (26), and looking at the matching 3-j symbols in (26) and the W-E theorem,

\[ \langle L' || Y_k || L \rangle = (-1)^L \sqrt{\frac{(2L'+1)(2L+1)(2k+1)}{4\pi}} \begin{pmatrix} L' & k & L \\ 0 & 0 & 0 \end{pmatrix} \]

or

\[ \langle L' || C(k) || L \rangle = (-1)^L \sqrt{\frac{(2L'+1)(2L+1)}{4\pi}} \begin{pmatrix} L' & k & L \\ 0 & 0 & 0 \end{pmatrix} \]

(b) \[ \langle j || f || j' \rangle \]

Take the \( J_0 \) component --

\[ \langle j_m | J_0 | j_{m'} \rangle = (-1)^{j-m} \begin{pmatrix} j & m' \\ -m & m \end{pmatrix} \langle j || f || j' \rangle \]

\[ = m \delta_{jj'} \delta_{mm'} \frac{i}{\sqrt{2j+1}} \]

\[ = (-1)^{j-m} \frac{m}{\sqrt{(2j+1)(j+1)}} \delta_{mm'} \]

\[ \Rightarrow \langle j || f || j' \rangle = \frac{i}{\sqrt{2j+1}} \delta_{jj'} \]

(c) \[ \langle \frac{1}{2} || \frac{3}{2} || \frac{1}{2} \rangle = \frac{1}{\sqrt{3}} \]

(d) \[ \langle j || 1 || j' \rangle = \sqrt{2j+1} \delta_{jj'} \]
(1) Scalar product of 2 commuting tensor operators:

Suppose \( T(k) \) operates on part 1 of a coupled system, with quantum numbers \( j_1, m_1 \).

And suppose \( U(k) \) operates on part 2 \( (j_2, m_2) \).

Then, \( <\alpha_{j_1}j_2jm | T(k), U(k) | \alpha'_{j_1'}j_2'j_2'm> \)

\[ = (-1)^{j_1+j_2+j_1'} \sqrt{j_1j_2j_1'j_2'} \sum_{m_1m_2} \{ j_1' j_2' j_1 j_2 \} \]

\[ \times \sum_{\alpha''} <\alpha_{j_1}|| T(k) || \alpha'_{j_1'}><\alpha''_{j_2}|| U(k) || \alpha'_{j_2'}> \]

(2) Matrix elements of \( T(k) \) operating on part 1 in coupled scheme:

The U-E theorem gives \( m, m', q \) dependence. So we specify the reduced matrix element,

\[ <\alpha_{j_1}j_2jm | T(q) | \alpha'_{j_1'}j_2'j_2'm> \]

\[ = (-1)^{j-m} \left( \begin{array}{ccc} j & k & j_1' \\ -m & q & m' \end{array} \right) \sum_{\alpha''} <\alpha_{j_1}j_2jm || T(k)|| \alpha'_{j_1'}j_2'j_2'm> \]

And the theorem gives its value:

\[ <\alpha_{j_1}j_2jm || T(k)|| \alpha'_{j_1'}j_2'j_2'm> \]

\[ = (-1)^{j_1+j_2+j_1'+(j_1')} \sqrt{(2j+m)(2j'+m)} \left\{ j_1' j_2' j_1 j_2 \right\} \]

\[ \times <\alpha_{j_1}|| T(k)|| \alpha'_{j_1'}> \]

(3) For \( U(k) \) operating on part 2, take (35) with \( j_1 \rightarrow j_2', j_2 \rightarrow j_1, j_1' \rightarrow j_2', j_2' \rightarrow j_1' \), except phase factor is \( (-1)^{j_1+j_2'+j+k} \).