Propagation of Gaussian Beams in Homogeneous Media

Physics 4150
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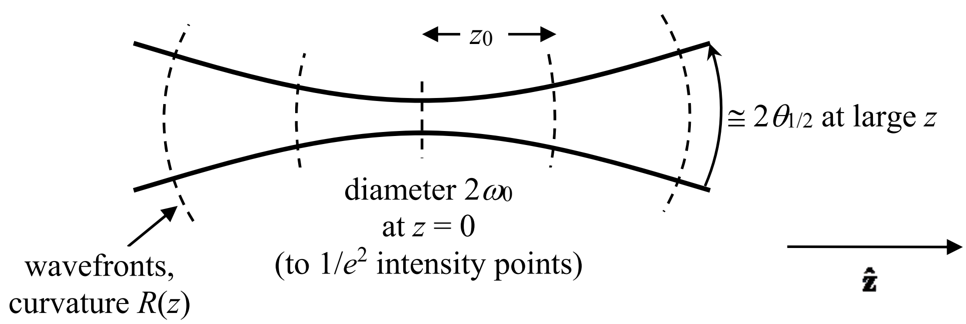
Even though a beam with a Gaussian transverse intensity profile is not an exact solution to the wave equation, it is a very widely used approximation because it provides a good description of the output of many lasers. As the beam propagates it retains a Gaussian cross-section, but its size changes due to diffraction as shown in the sketch below. The sketch also introduces some of the parameters we will use in this discussion. The best way to solve for the effects of diffraction on such a beam is to work directly with the wave equation, rather than to use approximate methods that fail near the origin, $z = 0$. There is a brief summary of the most useful equations describing Gaussian beams in Hecht, Section 13.1 (pp. 594-596). However, he does not describe the underlying theory. Here I give a brief account based loosely on Yariv and Yeh, *Photonics (6th Ed.)* and on Pedrotti[3], *Optics*.



A. Approximate wave equation

We are interested primarily in solutions that have cylindrical symmetry, so we will look for approximate solutions to the wave equation where the electric field has the form

$$\tilde{\mathbf{E}} = \mathbf{E_0} f(r,z) e^{i(kz-\omega t)}, \qquad (1)$$

where $f(r,z)$ describes the transverse profile of the laser. We will assume that the beam diameter changes with z only on a scale much larger than the wavelength λ. This assumption of a *slowly varying amplitude* is the key approximation that is required.

Assuming that the medium is homogeneous and has no free charges, the wave equation for the electric field is

$$\nabla^2 \tilde{\mathbf{E}} = \frac{1}{v^2}\frac{\partial^2 \tilde{\mathbf{E}}}{\partial t^2}. \qquad (2)$$

Given the harmonic time dependence of Eq. (1), this simplifies immediately to the *Helmholtz equation*,

$$\nabla^2 \tilde{\mathbf{E}} + k^2 \tilde{\mathbf{E}} = 0. \qquad (3)$$

The next step is to write out the spatial derivatives explictly,

$$\nabla^2\tilde{\mathbf{E}} = \mathbf{E}_0\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}+2ik\frac{\delta f}{\delta z}-k^2 f\right)e^{i(\omega t-kz)} \cong \mathbf{E}_0\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+2ik\frac{\delta f}{\delta z}-k^2 f\right)e^{i(\omega t-kz)}.$$

In the right-hand expression the second derivative with respect to z has been dropped because of the slowly varying envelope approximation, which can be stated mathematically as $\partial f/\partial z << k$. Upon inserting the resulting expression into Eq. (3) the two k^2 terms cancel, leaving a fairly simple differential equation for f:

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+2ik\frac{\delta f}{\delta z}=0, \text{ or in cylindrical coordinates, } \frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}+2ik\frac{\delta f}{\delta z}=0. \quad (4)$$

Now we look for solutions with a Gaussian transverse profile, which we write by introducing two complex functions $P(z)$ and $q(z)$ to define the longitudinal variation of the phase and the radius of curvature, with seemingly arbitrary multiplicative factors chosen for our later convenience:

$$f(r,z)=\exp\left[i\left(P(z)+\frac{kr^2}{2q(z)}\right)\right]. \quad (5)$$

Substituting this into Eq. (4), we find a scalar term and a term in r^2:

$$\frac{2ik}{q}-2k\frac{\partial P}{\partial z}+\left(\frac{k^2}{q^2}\frac{\delta q}{\delta z}-\frac{k^2}{q^2}\right)r^2 = 0. \quad (6)$$

For this to be a solution for all r, the scalar term and the quadratic term must equal zero separately. Thus we are finally left with a pair of equations for P and q,

$$\frac{\delta q}{\delta z}=1 \text{ and } \frac{\partial P}{\partial z}=\frac{i}{q}, \text{ which are easily solved to yield}$$
$$q(z)=q_0+z \text{ and } P(z)=i\ln\left(1+\frac{z}{q_0}\right). \quad (7)$$

The final step is to note that if we write q_0 in terms of real and imaginary parts, $q_{0r}+iq_{0i}$, it is possible to shift the origin so that at z=0, q_0 is purely imaginary. This corresponds to placing the origin at the minimum-diameter "beam waist". Further, the imaginary part of q_0 must be negative in order for $f(r,z)$ to vanish as $r\to\infty$, so for this choice of the origin we can write

$$q(z)=z-iz_0 \text{ and } P(z)=i\ln\left(1+\frac{iz}{z_0}\right). \quad (8)$$

B. Properties of Gaussian Beams

The formal solution for a Gaussian beam is already given by Eqs. (1), (5), and (8), but to gain some physical insight it is very helpful to define two real functions $R(z)$ and $\omega(z)$ to replace the complex function $q(z)$, with some multiplicative factors again chosen for later convenience:

$$\frac{1}{q(z)}=\frac{1}{z-iz_0}=\frac{1}{R(z)}+i\frac{\lambda}{\pi n\omega^2(z)}. \quad (9)$$

Equating the real and imaginary parts of Eq. (9) yields the z-dependence of R and ω,

$$R(z) = z\left(1+\frac{z_0^2}{z^2}\right) \text{ and}$$

$$\omega^2(z) = \omega_0^2\left(1+\frac{z^2}{z_0^2}\right), \text{ where} \tag{10}$$

$$\omega_0 \equiv \sqrt{\frac{\lambda z_0}{n\pi}}.$$

The parameter z_0 is called the *confocal parameter*, and ω_0 is the *beam waist* radius. If either is known, the other can be determined from the final relation in Eq. (10). Physically the *spot size* of the Gaussian beam is given by $\omega(z)$, and the *radius of curvature* of the wavefronts by $R(z)$, as shown in the sketch above. The interpretation of ω as the spot size can be justified mathematically by rewriting the full electric field of Eq. (1) in terms of these functions,

$$\tilde{\mathbf{E}} = \mathbf{E_0}\frac{\omega_0}{\omega(z)}e^{-r^2/\omega^2(z)}e^{ikr^2/2R(z)}e^{-i\tan^{-1}(z/z_0)}e^{i(kz-\omega t)}. \tag{11}$$

To understand the curvature, note that for $z \gg r$, with $r = \sqrt{x^2+y^2}$,

$$\frac{1}{R}e^{ikR} = \frac{1}{R}e^{ik\sqrt{x^2+y^2+z^2}} \cong e^{ikz+ikr^2/(2R)}. \tag{12}$$

At position z the irradiance is

$$I = \frac{1}{2}\varepsilon_0 nc|E|^2 = I_0\left(\frac{\omega_0}{\omega(z)}\right)^2 e^{-2r^2/\omega^2(z)}. \tag{13}$$

Thus the spot size $\omega(z)$ corresponds to the $1/e^2$ radius of the beam (measured in terms of power, not electric field). The smallest value that it ever attains is at $z = 0$, where the spot size is equal to the beam waist parameter, ω_0. Away from this location the beam spreads quadratically according to Eq. (10). The confocal parameter z_0 defines the scale for this spreading: at $z = z_0$ the spot size is $\omega = \sqrt{2}\,\omega_0$, and the peak irradiance is reduced by a factor of two.

As sketched in the figure, the radius of curvature $R(z)$ of the wavefronts is infinite at $z = 0$, meaning that the wavefronts are flat and perpendicular to the z axis. Further away, the radius of curvature gradually increases, approaching the limit $R = z$ that would characterize a pure spherical wave. It is also easy to show that at large z, the half-angle describing the beam divergence measured to the $1/e^2$ point approaches the value

$$\theta_{1/2} = \frac{\lambda}{\pi\omega_0 n}. \tag{14}$$

This limit where $z \gg z_0$ is the region in which ordinary geometric optics can be used with good accuracy.

We have not discussed the focusing of a Gaussian beam by a lens. If the input beam has a waist at the lens, the equations are not difficult. They have been incorporated into the "Topticalc" optics calculator program from Toptica, which is recommended as a quick way to evaluate the spot size of a focused beam. For the more general case of an input beam with an arbitrary waist position, there is a matrix formalism for describing the propagation of Gaussian beams, including their focusing by multiple lenses and other optical elements. You can read about this method, often called the "ABCD law," in most advanced texts on optics or lasers. It is also implemented in most modern ray-tracing programs, including the OSLO program that we use in Physics 4150.

It's also important to note that for optical beams in a resonator, such as the curved-mirror cavities used for many simple lasers, the Gaussian beam is only the lowest-order solution. A more general treatment allowing for the possibility of azimuthal variations in the field reveals that the modes of a resonator can be written in the slowly-varying amplitude approximation as Gauss-Hermite polynomials. The lowest-order mode, often designated TEM_{00}, is identical to the Gaussian beam. An important task in designing a practical laser is to suppress all of the higher-order transverse modes so that the output is a relatively "clean" Gaussian mode. Similar considerations arise in the design of fiber optics.

Example: Spreading of a He-Ne laser beam

Consider the beam from a 632 nm He-Ne laser. If it has a beam waist located at the laser output with a $1/e^2$ radius of $\omega_0 = 0.5$ mm, the confocal parameter is

$$z_0 = \frac{\pi\omega_0^{\,2}}{\lambda} = 1.24 \text{ m.}$$

From Eq. (10) we find that the beam doubles in size, $\omega = 2\omega_0 = 1$ mm, when $z^2/z_0^2 = 3$, or $z =$ 2.15 m. At a distance of several meters the beam diverges like a bundle of rays from a point source, with a half angle of

$$\theta_{1/2} = \frac{\lambda}{\pi\omega_0} = 0.402 \text{ mrad.}$$

At a distance of 10 m from the laser, we can estimate from the divergence that the radius is approximately 10(0.402) = 4.02 mm, while from Eq. (10) we find the exact solution, which differs very little:

$$\omega(10 \text{ m}) = \omega_0\sqrt{1 + \frac{z^2}{z_0^{\,2}}} = 4.06 \text{ mm.}$$