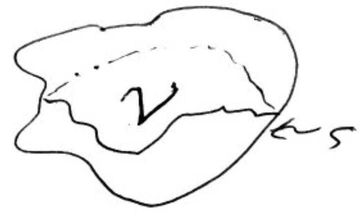
IV. Uniqueness theorem (simplified version)

⇒ The solution to Laplace's equation in a volume $\mathcal{V}$ is uniquely determined if $V(\vec{r})$ is specified everywhere on a bounding surface S

Proof: Assume V_1 and V_2 are both solutions,



$\nabla^2 V_1 = 0$ and $\nabla^2 V_2 = 0$.

Consider the difference $V_3 = V_1 - V_2$. Then

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0$$

At the boundary S, both V_1 and V_2 must satisfy the boundary conditions, so $V_1 = V_2$ on S.

Thus, 1) V_3 satisfies Laplace's eqn.

2) $V_3 = 0$ everywhere on S.

But V_3 has no extrema except at boundaries

⇒ it's zero everywhere in $\mathcal{V}$.

a) Note that S can go to ∞ (usually define $V=0$ there).

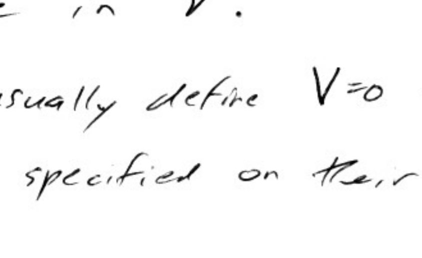
b) Also, "islands" are OK if V specified on their surface, too

c) Also works for Poisson's equation if $\rho(\vec{r})$ is specified everywhere:

$$\nabla^2 V_3 = -\frac{\rho}{\varepsilon_0} + \frac{\rho}{\varepsilon_0} = 0$$

Ex: Charge-free cavity in a conductor:

1) V = constant on wall, V_0 (it's a conductor!)



2) Clearly, a solution is $V = V_0$ everywhere inside.

3) Uniqueness ⇒ only solution is $V = V_0$. (Can also prove because no local maxima, minima.)

4) Thus $\vec{E} = -\vec{\nabla} V = 0$ (alternate to earlier proof.)

IV. A <u>Uniqueness theorem</u> (full version, both equations)

Assume a volume $\mathcal{V}$ is bounded by surface S and has a specified charge density $\rho(\vec{r})$. What must be specified so $\vec{E}(\vec{r})$ is uniquely determined? (i.e., $V(\vec{r})$ is known to within a single additive constant.)

Assume $V_1(\vec{r})$, $V_2(\vec{r})$ are solutions,

look at $V_3 = V_1 - V_2$.

Then $\nabla^2 V_3 = -\frac{\rho}{\varepsilon_0} + \frac{\rho}{\varepsilon_0} = 0$

We want to know what is required to force the condition $V_3 = k$, k a scalar constant.

i.e., $\vec{\nabla} V_3 = 0$ everywhere.

$$\text{Well,} \quad \underset{\text{(everywhere)}}{\vec{\nabla} V_3 = 0} \iff \int_{\mathcal{V}} (\vec{\nabla} V_3)^2 \, d\tau = 0 \quad \left(= \int E_3^2 \, d\tau\right)$$

Rewrite to take advantage of divergence theorem —

$$\vec{\nabla}\cdot(V_3 \vec{\nabla} V_3) = (\vec{\nabla} V_3)^2 + V_3 \underbrace{\nabla^2 V_3}_{=0}$$

So it will suffice to obtain

$$\int_{\mathcal{V}} \vec{\nabla}\cdot(V_3 \vec{\nabla} V_3)\, d\tau = 0$$

$$\text{or} \quad \int_S (V_3 \vec{\nabla} V_3)\cdot d\vec{a} = 0 = \int (V_3 \vec{\nabla} V_3)\cdot \hat{n}\, da$$

This will be the case if at each point on S we have either

$$\begin{cases} V_3 = 0 & (V_1 = V_2 \text{ on } S) \\ \underline{\text{or}} \quad \hat{n}\cdot\vec{\nabla} V_3 = 0 & (\hat{n}\cdot\vec{\nabla} V_1 = \hat{n}\cdot\nabla V_2 \text{ on } S) \end{cases}$$

Thus the field $\vec{E}(\vec{r})$ is uniquely determined (and V to within a constant) if for each point on S, V satisfies either the <u>Dirichlet boundary condition</u>,

$V(\vec{r})$ specified on S, (A)

or the <u>Neumann boundary condition</u>,

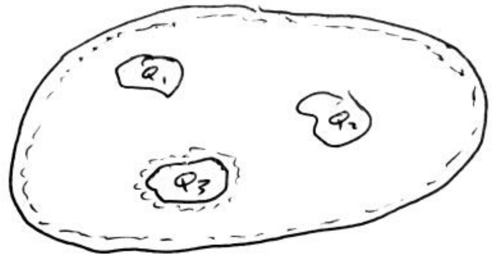
$\hat{n}\cdot\vec{E}$ $(\equiv E^{\perp})$ is specified on S (B)

If (A) is specified <u>anywhere</u> we also know the additive constant in V.

V. "Second uniqueness theorem"

In a volume $\mathcal{V}$ containing only conductors and a specified charge density $\rho(\vec{r})$, $\vec{E}$ is uniquely determined if the total charge on each conductor is given. (Overall boundary is one of the conductors or at ∞.)

Let $\vec{E}_1$, $\vec{E}_2$ be two solutions.



For a Gaussian surface around each conductor i,

$$\oint_{S_i} \vec{E}_1 \cdot d\vec{a} = \frac{Q_i}{\varepsilon_0} = \oint_{S_i} \vec{E}_2 \cdot d\vec{a} \qquad \text{(Gauss' law)}$$

Also $$\oint_{\substack{\text{outer}\\ \text{boundary}}} \vec{E}_1 \cdot d\vec{a} = \frac{Q_{tot}}{\varepsilon_0} = \oint \vec{E}_2 \cdot d\vec{a} = \frac{\sum_i Q_i}{\varepsilon_0} \leftarrow \text{all conductors}$$

Define $\vec{E}_3 = \vec{E}_1 - \vec{E}_2$. Then $$\oint_{S_i \text{ or outer bdry}} \vec{E}_3 \cdot d\vec{a} = \frac{Q_i}{\varepsilon_0} - \frac{Q_i}{\varepsilon_0} = 0 \quad (1)$$

In the volume between conductors, $\vec{\nabla} \cdot \vec{E}_3 = 0$,

since $\vec{\nabla} \cdot \vec{E}_1 - \vec{\nabla} \cdot \vec{E}_2 = +\frac{\rho}{\varepsilon_0} - \frac{\rho}{\varepsilon_0} = 0$

Finally, $V_3 = V_1 - V_2 =$ a constant V_{3i} on each conductor.

Trick: (as before) $$\vec{\nabla} \cdot (V_3 \vec{E}_3) = V_3 \underbrace{(\vec{\nabla} \cdot E_3)}_{=0} + \vec{E}_3 \cdot \vec{\nabla} V_3 = -\vec{E}_3 \cdot \vec{E}_3 = -E_3^2$$

If we integrate over all the open space,

$$\int_{\substack{\text{Volume}\\ \text{between conductors}}} (\vec{\nabla} \cdot V_3 \vec{E}_3)\, d\tau \underset{\text{(div. thm.)}}{=} \oint_{\substack{\text{all}\\ \text{surfaces}}} V_3 \vec{E}_3 \cdot d\vec{a} = -\int E_3^2\, d\tau$$

(surfaces ($d\vec{a}$ now inwards))

$$= \sum_i V_{3i} \underbrace{\oint_{S_i} \vec{E}_3 \cdot d\vec{a}}_{\text{see (1) (above)}}, \quad \text{so} \quad \int E_3^2\, d\tau = 0$$

constant (see above) $\quad = 0$

$\Rightarrow \vec{E}_3(\vec{r}) = 0$

or $\boxed{\vec{E}_1 = \vec{E}_2}$

It follows that if $\begin{cases} Q_i \to 2Q_i \\ \rho \to 2\rho \end{cases}$, $\vec{E} \to 2\vec{E}$ is only solution.

This is essentially a proof that specifying Q on conductor is equivalent to specifying Neumann boundary conditions. The distribution of σ on the surface has a unique solution, which means $E^{\perp} = \sigma/\varepsilon_0$ is known on the surface.