III. Volume integrals

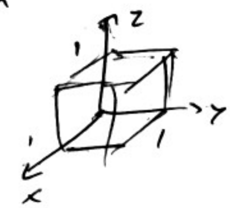
$$\int_V \Phi(\vec{r})\, d\tau$$

$d\tau$ ← volume element, $dx\,dy\,dz$

$\Phi(\vec{r})$ ← scalar function

Example: $\Phi = x^2 y z^2$

$V =$ a unit cube, one corner at origin



$$\int \Phi\, d\tau$$

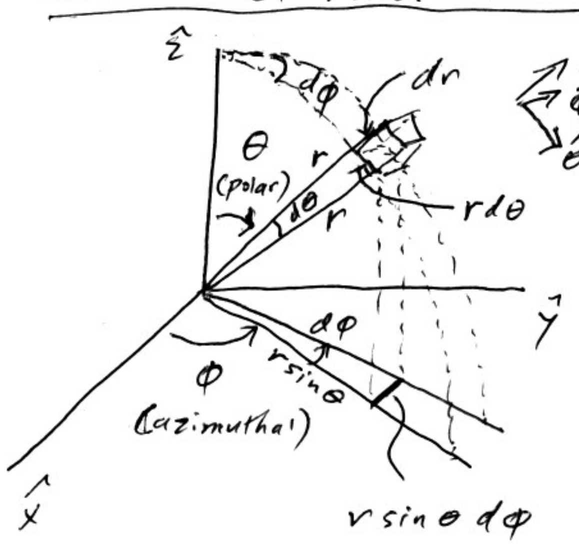
$$= \int_0^1 dx \int_0^1 dy \int_0^1 dz\; x^2 y z^2$$

$$= \int_0^1 x^2 dx \left(\int_0^1 y\, dy \left(\int_0^1 z^2 dz \right) \right) = \left(\int_0^1 x^2 dx \right) \left(\underbrace{\int_0^1 y\, dy}_{\text{indep of } x} \right) \left(\underbrace{\int_0^1 z^2 dz}_{\text{indep. of } x,y} \right)$$

$$= \frac{x^3}{3}\Big|_0^1 \; \frac{y^2}{2}\Big|_0^1 \; \frac{z^3}{3}\Big|_0^1 = \frac{1}{18}$$

In general, it's hard to incorporate the boundary shape into the limits of integration. We often need $d\tau$ in spherical coordinates:

Volume element in spherical coordinates (see 1.4.1)



For the small box,

$$d\tau = dr\,(r\,d\theta)(r\sin\theta\, d\phi)$$

$$\boxed{d\tau = r^2 \sin\theta\, d\theta\, d\phi\, dr}$$

$\theta: 0 \to \pi$

$\phi: 0 \to 2\pi$

Solid angle

If we are interest only in θ and ϕ, define

$$d\Omega = \frac{d\tau}{r^2 dr} = \sin\theta \, d\theta \, d\phi$$

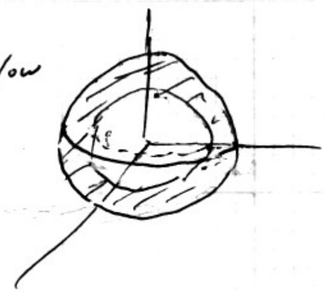
Note that

$$\int_{\text{all } \theta, \phi} d\Omega = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \sin\theta \, d\theta \, d\phi$$

$$= -\cos\theta \Big|_0^{\pi} (2\pi) = 4\pi \ \underline{\text{steradians}}$$

and obviously $d\tau = r^2 \, d\Omega \, dr$

Ex: volume of spherical shell (i.e., hollow shell) from r_1 to r_2



$$V = \int_{r_1}^{r_2} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} d\tau$$

$$= 2\pi \int_{r_1}^{r_2} \int_0^{\pi} r^2 \sin\theta \, d\theta \, dr$$

$$= 2\pi \left(-\cos\theta \Big|_0^{\pi}\right) \left(\frac{r^3}{3}\Big|_{r_1}^{r_2}\right)$$

$$= \frac{4\pi}{3}\left(r_2^3 - r_1^3\right), \text{ as expected}$$

This is really just $\left(\int_{\text{all angles}} d\Omega\right)\left(\int_{r_1}^{r_2} r^2 dr\right)$