Separation in Spherical Coordinates

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

We will separate this as,

$$V = R(r)\,\Theta(\theta)\,\Phi(\phi)$$

Substitute and multiply the result by $\frac{r^2}{R\Theta\Phi}$ to get

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0$$

We will look only at azimuthally symmetric problems where Φ = constant, so each of the other terms must be equal to a constant, with their sum $=0$. Call the constant $\ell(\ell+1)$:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \ell(\ell+1), \quad \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -\ell(\ell+1)$$

1) Radial equation:

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \ell(\ell+1)R \Rightarrow \boxed{R(r) = Ar^\ell + \frac{B}{r^{\ell+1}}}$$

A, B depend on b.c.'s.

2) Angular equation:

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -\ell(\ell+1)\sin\theta\,\Theta$$

This is a version of Legendre's equation, whose solutions (for $\cos\theta$) are Legendre polynomials",

$$\boxed{\Theta(\theta) = P_\ell(\cos\theta)}$$

They are given by Rodriguez' formula,

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell, \quad x = \cos\theta$$

So $P_0(x) = 1$

$P_1(x) = x \qquad (x = \cos\theta)$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

etc.

(Has only even powers if ℓ even, odd if ℓ odd).

Why not two independent solutions (since 2nd order)?

The other solutions diverge at $\theta = 0$ or π,

e.g. $\ln(\tan\frac{\theta}{2})$

The Legendre polynomials form a complete orthogonal set on the interval $(-1, 1)$:

$$\int_{-1}^{1} P_\ell(x)\, P_{\ell'}(x)\, dx = \frac{2}{2\ell+1}\delta_{\ell\ell'},$$

$$f(x) = \sum_{\ell=0}^{\infty} \cancel{A_\ell}\, c_\ell\, P_\ell(x) \qquad (\text{for } -1 \le x \le 1)$$

Summarizing, for any azimuthally symmetric problem,

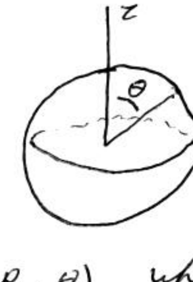
$$\boxed{V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)}$$

(absorb coeff. into A, B).

(more generally, have $e^{\pm im\phi}$ for azimuthal $\angle$)

Ex 1: $V_0(\theta)$ is specified on a spherical shell

find V inside (for $r < R$):



Need V finite as $r \to 0$ (no charges at origin) $\Rightarrow B_\ell = 0$,

$$\boxed{V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)} = V_0(R,\theta) \text{ when } r = R$$

Use "Fourier's trick" with the orthogonal Legendre polynomials: $dx = d(\cos\theta) = -\sin\theta\, d\theta$, so $$\int_0^\pi V_0 \, P_{\ell'}(\cos\theta) \sin\theta \, d\theta = \sum_{\ell=0}^{\infty} A_\ell R^\ell \underbrace{\int_0^\pi P_\ell(\cos\theta) P_{\ell'}(\cos\theta) \sin\theta \, d\theta}_{= \frac{2}{2\ell+1}\delta_{\ell\ell'}}$$

$$A_{\ell'} R^{\ell} \frac{2}{2\ell'+1} = \int_0^\pi V_0(\theta) P_{\ell'}(\cos\theta) \sin\theta \, d\theta$$

or $$\boxed{A_\ell = \frac{2\ell+1}{2R^\ell} \int_0^\pi V_0(\theta) P_\ell(\cos\theta) \sin\theta \, d\theta}$$

Integrals are best done by expanding V_0 in Legendre polynomials, then using orthogonality relations.

For example,

i) If $V_0(\theta) = V_1$ (constant)

$= V_1 P_0(\cos\theta)$, then

$$A_0 = \frac{V_1}{2}\left(\frac{2}{2(0)+1}\right) = V_1 \,, \quad \text{all others} = 0,$$

and $V = (V_1) r^0 P_0 = V_1$ (for $r < R$) ✓

2) If $V_0(\theta) = K \sin^2(\theta/2)$ (as in Griffiths)

$= \frac{K}{2}(1-\cos\theta) = \frac{K}{2}(P_0(\cos\theta) - P_1(\cos\theta))$,

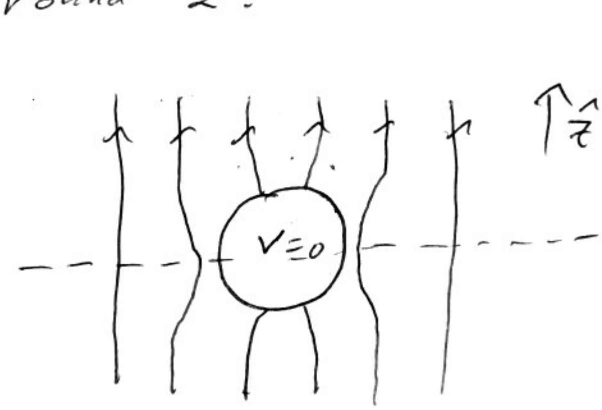
get $A_0 = \frac{K}{2}$, $A_1 = \frac{-K}{2R}$, all others $= 0$.

$$V(r,\theta) = \frac{K}{2} r^0 P_0(\cos\theta) - \frac{Kr}{2R} P_1(\cos\theta)$$
$$= \frac{K}{2}\left(1 - \frac{r}{R}\cos\theta\right)$$

<u>Note</u>: To find solutions for $r > R$, must set A_ℓ's to zero, and re-solve using the B_ℓ's.
See Griffiths Ex. 3.7 for more details.

Ex 2: Grounded conducting sphere in a uniform external field, round 2:

Note that x-y plane is an equipotential, by symmetry, so $V=0$ here.



So as $r \to \infty$, $\vec{E} \to E_0 \hat{z}$

and $V \to -E_0 z = -E_0 r \cos\theta$

$$= -E_0 r P_1(\cos\theta)$$

Thus the b.c.'s are

1) $V(r,\theta) \to -E_0 r P_1(x)$ as $r \to \infty$

2) $V(r,\theta) = 0$ on the sphere $(r=R)$ and for $z=0$ (x-y plane).

Starting with $V(r,\theta) = \sum_0^\infty \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$,

2) $\Rightarrow A_\ell R^\ell + \frac{B_\ell}{R^{\ell+1}} = 0$, so

$$B_\ell = -A_\ell R^{2\ell+1}$$

Now $V(r,\theta) = \sum A_\ell \left(r^\ell - \frac{R^{2\ell+1}}{r^{\ell+1}}\right) P_\ell(\cos\theta)$

1) $\Rightarrow$ as $r \to \infty$, $\sum_{\ell=0}^\infty A_\ell r^\ell P_\ell(\cos\theta) = -E_0 r P_1(\cos\theta)$

Thus $A_1 = -E_0$, all others $= 0$.

$$V(r,\theta) = -E_0 r P_1(\cos\theta) + E_0 \frac{R^3}{r^2} P_1 \cos\theta$$

$$= \underbrace{-E_0 r \cos\theta}_{\Rightarrow \text{ uniform field}} + \underbrace{E_0 \frac{R^3}{r^2} \cos\theta}_{\Rightarrow \text{ field of a point dipole, } p = 4\pi\varepsilon_0 R^3 E_0}$$

This matches perfectly the result on p. MI-5.

Note: can easily solve for induced surface charge areal density,

$$\sigma(\theta) = -\varepsilon_0 \frac{\partial V}{\partial n} = -\varepsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=R} \quad \text{(for a conductor)}$$

$$= \varepsilon_0 E_0 \cos\theta - \varepsilon_0 E_0 R^3 \frac{-2}{R^3} \cos\theta$$

$$= 3\varepsilon_0 E_0 \cos\theta \quad \text{(positive at top, neg. at bottom)}$$

Ex 2.5:

There's an additional example in Griffiths that shows a Neumann boundary condition — what if $\sigma(\theta)$ is given and not $V(\theta)$?

Then (not assuming a conductor)

1) V is continuous at a charge shell ($V_{in} = V_{out}$ at $r = R$)

2) $$\left.\left(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r}\right)\right|_{r=R} = -\frac{\sigma}{\varepsilon_0}$$

Griffiths' example is for a non-conducting shell. What if we instead solve the conducting sphere, "backwards"? Then only V_{out} is needed.

We still have $V \to -E_0 r P_1(\cos\theta)$ as $r \to \infty$,

So $A_1 = -E_0$ and all other A_ℓ's $= 0$. (a)

Also, if $\sigma(R,\theta) = 3\varepsilon_0 E_0 \cos\theta$, given $V = \text{const}$ inside a conductor,

$$\left.\frac{\partial V_{out}}{\partial r}\right|_{r=R} = -3E_0\cos\theta$$

$$\left.\frac{\partial}{\partial r}\left(-E_0 r P_1(\cos\theta) + \sum \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)\right)\right|_{r=R} = -3E_0\cos\theta$$

$$-E_0 P_1(\cos\theta) + \sum_\ell \frac{-(\ell+1) B_\ell}{R^{\ell+2}} P_\ell(\cos\theta) = -3 E_0 P_1(\cos\theta)$$

So only $\ell = 1$ appears, and

$$-E_0 P_1(\cos\theta) - \frac{2 B_1}{R^3} P_1(\cos\theta) = -3 E_0 P_1(\cos\theta)$$

$$\Rightarrow B_1 = E_0 R^3$$

and $V = -E_0 r \cos\theta + E_0 \frac{R^3}{r^2} \cos\theta$ as before.

<u>Notes</u>: if not a conductor, $V_{in} \neq$ constant

$\Rightarrow$ must equate V_{in}, V_{out} at $r = R$.

Also, if b.c. is not a simple sum of P_ℓ's, must use "Fourier's trick" to find A_ℓ, B_ℓ

(Ex: 

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