Separation of Variables for Laplace's equation

Concept:

1) Choose a coordinate system in which the boundaries are simple
2) Try to separate Laplace's equation,

$$V = V(\xi_1)\,V(\xi_2)\,V(\xi_3)$$

3) Expand V in a complete orthogonal series and find the coefficients to give a sum of separable solutions that give the full solution.

A complete orthogonal series is one that can be used to express any function as a linear combination,

$$f(y) = \sum_{n=1}^{\infty} c_n f_n(y),$$

and for which the integral of the product of any two different members is zero,

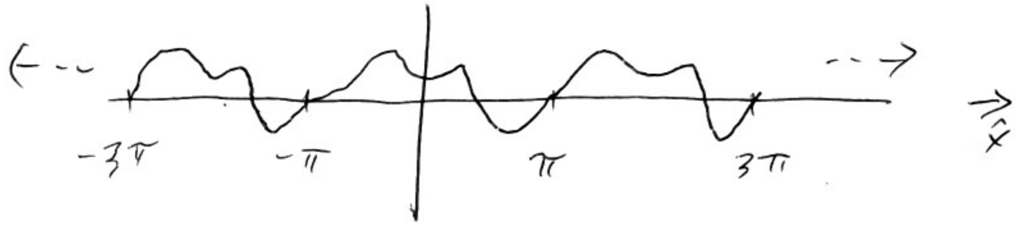
$$\int f_n(y)\, f_{n'}(y)\, dy = 0 \quad \text{if } n' \neq n$$

In Cartesian coordinates a good choice is the Fourier series of sines and cosines. Griffiths uses only real notation; I will use some complex notation.

Fourier series:

Use the notation $e^{i\theta} = \cos\theta + i\sin\theta$

Expand a periodic function on the interval $-\pi \cdots \pi$ (easily generalized to $-a \cdots a$):



Write $f(x) = \sum_{m=-\infty}^{\infty} c_m e^{-imx}$

(for $-a \cdots a$ use $e^{-im\pi\frac{x}{a}}$)

To determine the coefficients c_m, notice that

$$f(x)\,e^{im'x} = \sum_m c_m e^{i(m'-m)x}$$

So

$$\int_{-\pi}^{\pi} f(x)\,e^{im'x} = \sum_m c_m \underbrace{\int_{-\pi}^{\pi} e^{ix(m'-m)}\,dx}_{\substack{= 2\pi \text{ if } m'=m \\ 0 \text{ otherwise}}}$$

$$= 2\pi c_m'$$

Thus $\boxed{c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{imx}\,dx}$

If $f(x)$ is real the coefficients come in pairs; arranging as sines and cosines and extending the period to $-a \cdots a$ yields

$$\boxed{\begin{aligned} f(x) &= c_0 + \sum_{n=1}^{\infty}\left(c_n \cos n\pi\frac{x}{a} + d_n \sin n\pi\frac{x}{a}\right) \\ c_0 &= \frac{1}{2a}\int_{-a}^{a} f(x)\,dx, \quad c_n = \frac{1}{a}\int_{-a}^{a} f(x)\cos\left(n\pi\frac{x}{a}\right)dx \\ d_n &= \frac{1}{a}\int_{-a}^{a} f(x)\sin\left(n\pi\frac{x}{a}\right)dx \end{aligned}}$$

"Periodic" potentials in Cartesian coordinates

can be done by continuation beyond boundary

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Writing $V = X(x)\,Y(y)\,Z(z)$,

$$\frac{d^2X}{dx^2} YZ + \frac{d^2Y}{dy^2} XZ + \frac{d^2Z}{dz^2} XY = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

Since X, Y, and Z can vary independently, works in general only if

$$\left.\begin{aligned} \frac{1}{X}\frac{d^2X}{dx^2} &= \text{const} = C_1 \\ \frac{1}{Y}\frac{d^2Y}{dy^2} &= C_2 \\ \frac{1}{Z}\frac{d^2Z}{dz^2} &= C_3 \end{aligned}\right\} \text{ where } C_1 + C_2 + C_3 = 0$$

It's convenient to write $C_1 = \alpha^2$, $C_2 = \beta^2$, $C_3 = \gamma^2$, but note at least one of these α, β, γ must be complex for Σ squares $= 0$.

The resulting equations have the form

$$\frac{d^2X}{dx^2} - \alpha^2 X = 0 \quad \text{, etc.}$$

e.g., if α imaginary, $\alpha = ik$,

$$\frac{d^2X}{dx^2} + k^2 X = 0 \qquad \text{(harmonic oscillator)}$$

$$X(x) = \mathrm{Re}\left(Ae^{ikx} + Be^{-ikx}\right)$$

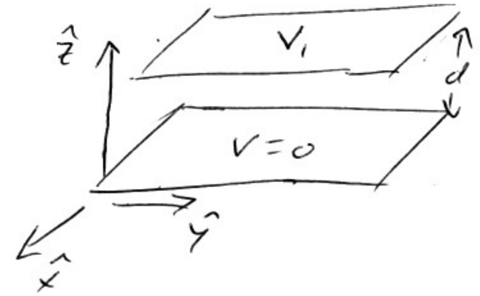
If α, β imaginary, γ real, so

$$Z(z) = Ce^{\gamma z} + De^{-\gamma z}$$

$\Rightarrow$ Solutions are a mix of exponentials and oscillatory terms.

For a particular problem, may need to sum solutions for a large set of α's, β's and γ's to meet the boundary conditions — for oscillatory soln's this is a Fourier series.

Ex 1: (trivial) Infinite capacitor plates



By symmetry,
$X(x) = Y(y) = \text{constant}.$

So $\alpha = \beta = 0 \Rightarrow \gamma = 0$, and $\frac{d^2 Z}{dz^2} = 0$, forcing

$$Z(z) = A + Bz.$$

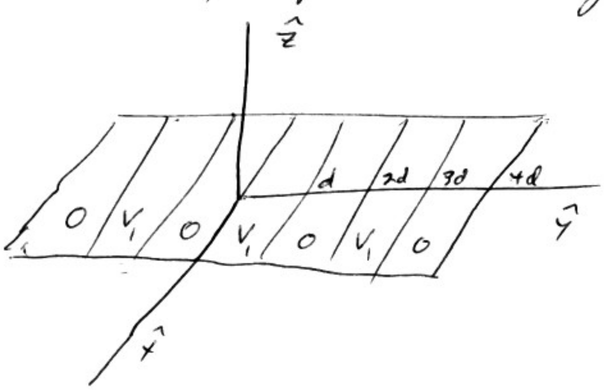
Thus $V = A' + B'z$

Boundary cond's: 1) At $z=0$ $V=0 \Rightarrow A'=0$

2) At $z=d$ $V=V_1 \Rightarrow B' = \frac{V_1}{d}$

Finally, $V = \frac{V_1}{d} z$, $\vec{E} = -\vec{\nabla} V = -\frac{V_1}{d}\hat{z}$

Ex 2: Explicitly oscillatory behavior: periodically poled charged sheet



1) Infinite along $\hat{x} \Rightarrow X(x) = \text{constant}$, $\alpha = 0$

2) $V \to \text{const.}$ as $z \to \infty$ or $-\infty$

(we don't really know if it's zero or not!)

OK, let's get started:

a) $\alpha = 0 \Rightarrow \beta^2 + \gamma^2 = 0$

b) Oscillatory in y with period $2d$

$\Rightarrow \beta$ is imaginary, solutions are sinusoidal

c) Odd function except for dc offset of $V_1/2$, so only the sin terms need be kept, and for $\beta = im\left(\frac{\pi}{d}\right)$, can write

$$Y_m(y) = c_m \sin\left(m \frac{2\pi}{2d} y\right)$$

modify m here, to make periodic on $-d \ldots d$, so $\beta = im\frac{\pi}{d}$ now.

(except Y_0 = constant)

d) γ is real, $\gamma = \pm i\beta$ for each m

So $$V(x,y,z) = V_0 + \sum_{m=1}^{\infty} c_m \sin\frac{m\pi y}{d}\left(D_m e^{m\pi\frac{z}{d}} + E_m e^{-m\pi\frac{z}{d}}\right)$$

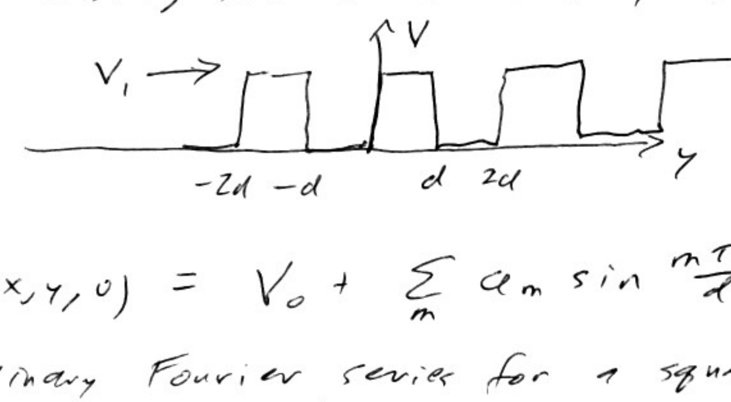
e) To obtain $V \to$ const as $|z| \to \infty$,

need $D_m = 0$ for $z > 0$, $E_m = 0$ for $z < 0$

Writing $c_m D_m = a_m$, $z<0$; $c_m E_m = a_m$, $z > 0$,

$$V(x,y,z) = V_0 + \sum_m a_m \sin\frac{m\pi y}{d} e^{-\frac{m\pi|z|}{d}}$$

f) Evaluate V_0, a_m using bdry conditions at $z=0$ (this could have been done before looking at z, if we wished, since it is a bdry at z = constant).



$$V(x,y,0) = V_0 + \sum_m a_m \sin\frac{m\pi y}{d}$$

Ordinary Fourier series for a square wave, period $2d$.

So, $$a_m = \frac{1}{d}\int_{-d}^{d} f(x) \sin\left(m\frac{\pi}{d}x\right) dx$$

Odd function $\Rightarrow b_m = 0$ (no cosines)

Since $V=0$ from $-d \to 0$, V_1 from $0 \to d$,

$$a_m = \frac{1}{d}\int_0^d V_1 \sin\left(m\pi \frac{y}{d}\right)$$

$$= \frac{V_1}{d}\left(\frac{-1}{\frac{m\pi}{d}}\right)\cos\frac{m\pi y}{d}\Big|_0^d$$

$$= -\frac{V_1}{m\pi}\left(\underbrace{\cos(m\pi) - 1}_{\text{zero if } m \text{ even, } -2 \text{ if odd}}\right) = \frac{2V_1}{m\pi}, \quad m \text{ odd.}$$

To get V_0, note
$$a_0 = \frac{1}{2d}\int_{-d}^{d} f(y)\,dy$$
$$= \frac{1}{2d}\int_0^d V_1\,dy = \frac{V_1}{2} \quad (= \text{avg. value})$$

Finally,

$$\boxed{V(x,y,z) = \frac{V_1}{2} + \frac{2V_1}{\pi}\sum_{m\ \text{odd}} \frac{1}{m}\sin\frac{m\pi y}{d}\, e^{-\frac{m\pi|z|}{d}}}$$

Long range:

If $|z| \gg d/\pi$, 1st term $\propto e^{-\frac{\pi|z|}{d}}$,
$m=3 \propto e^{-\frac{3\pi|z|}{d}}$, etc.

and $V \simeq \frac{V_1}{2} + \frac{2V_1}{\pi}\sin\frac{\pi y}{d}\, e^{-\frac{\pi|z|}{d}} + \cdots$

At $y = \pm nd$, get $V = \frac{V_1}{2}$ exactly, for all x, z.

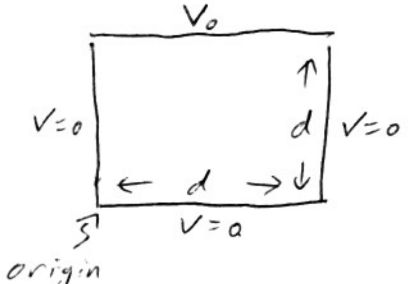
Quicker method:

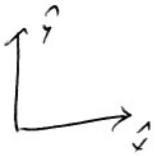
Could have set $Y(y) = V(x,y,0)$, then find z dependence.

OK since $V(x,y,0) = Z(0)\,Y(y) = \text{constant} \times Y(y)$.

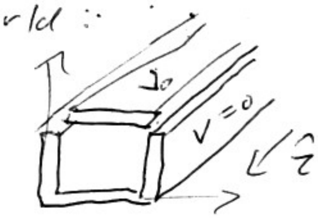
$\Rightarrow$ See Mathcad plots for various m's.

Ex 3: Same problem as in example numerical solution, essentially:





(Actually a section of a pipe in the real 3-D world:



)

1) Because $Z(z) = \text{constant}$, $\gamma = 0$, and $\alpha^2 + \beta^2 = 0$

2) $X(0) = X(d) = 0$, so can treat as periodic in x with period $2d$. Let's see how this works...

Since we can extend the solution along x as an odd function, we can write it as a sine series, with index m: to give zero at $x=0$, $x=d$

$$X(x) = A_m \sin \frac{m\pi x}{d} \qquad \left(\text{i.e., } \alpha = \frac{m\pi}{d} i\right)$$

Then $\beta = \pm \frac{m\pi}{d}$, and

$$Y(y) = B_m e^{\frac{m\pi y}{d}} + C_m e^{-\frac{m\pi y}{d}}$$

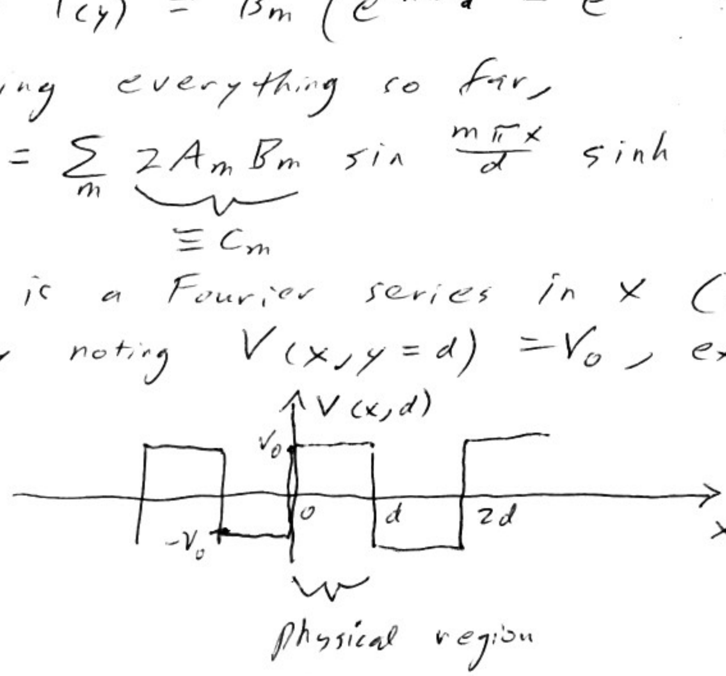
Because $V(x, y=0) = 0$, $B_m + C_m = 0$

$$\text{so } Y(y) = B_m \left(e^{m\pi \frac{y}{d}} - e^{-m\pi \frac{y}{d}}\right) = 2B_m \sinh \frac{m\pi y}{d}$$

Collecting everything so far,

$$V = \sum_m \underbrace{2A_m B_m}_{\equiv C_m} \sin \frac{m\pi x}{d} \sinh \frac{m\pi y}{d}$$

This is a Fourier series in x (again). Evaluate by noting $V(x, y=d) = V_0$, extending as odd fcn:



(could also extend as even fcn!)

Evaluate the coefficients as on p. SV – 2:

$C_0 = \overline{V} = 0$ over a full period

$$\underbrace{C_m \sinh\left(\frac{m\pi y}{d}\right)}_{\text{plays role of } C_m \text{ on p. SV-2}} \leftarrow \text{at } y=d \quad = \frac{1}{d}\left(\int_{-d}^{0} -V_0 \sin\frac{m\pi x}{d}\,dx + \int_{0}^{d} V_0 \sin\frac{m\pi x}{d}\,dx\right)$$

$$= \frac{V_0}{d}\frac{d}{m\pi}\left(\cos\frac{m\pi x}{d}\Big|_{-d}^{0} - \cos\frac{m\pi x}{d}\Big|_{0}^{d}\right)$$

$$C_m \sinh(m\pi) = \frac{V_0}{m\pi}\left(1 - \cos m\pi + 1 - \cos m\pi\right)$$

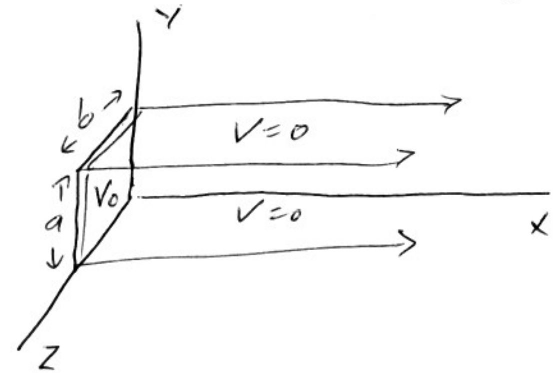
$$= \frac{V_0}{m\pi} \times \begin{cases} 4 & \text{if } m \text{ odd} \\ 0 & \text{if } m \text{ even} \end{cases}$$

The result is thus

$$V(x,y,z) = \frac{4V_0}{\pi}\sum_{m=1,3,5,7\cdots} \frac{1}{m\sinh(m\pi)} \sin\frac{m\pi x}{d} \sinh\frac{m\pi y}{d}$$

(can easily plot and compare to numerical solutions)

Ex 4: (Griffiths ex. 3.5) A true 3D problem:



Rectangular pipe,

1) $V = 0$ at $y = 0$
2) $y = a$
3) $z = 0$
4) $z = b$
5) $V \to 0$ as $x \to \infty$
6) $V = V_0$ at $x = 0$ (left face)

Solutions along y and z look like x in previous problem. Setting periods to $2a$ and $2b$, can write

$$Y(y) = C_m \sin\frac{m\pi y}{a}, \qquad Z(z) = d_n \sin\frac{n\pi z}{b}$$

Since $\beta = \frac{m\pi}{a} i$, $\gamma = \frac{n\pi}{b} i$, $\alpha^2 + \beta^2 + \gamma^2 = 0$

$$\Rightarrow \alpha_{mn} = \pm\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = \pm\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

If we write the two soln's as $\pm\alpha_{mn}$,
then $X(x) = a_{mn} e^{\alpha_{mn} x} + b_{mn} e^{-\alpha_{mn} x}$

Apply b.c. 5): As $x \to \infty$, $X(x) \to a_{mn} e^{\alpha_{mn} x}$

$\Rightarrow a_{mn} = 0$

So far we have found

$$V = \sum_m \sum_n \underbrace{b_{mn} c_m d_n}_{\equiv C_{m,n}} e^{-\alpha_{mn} x} \sin \frac{m\pi y}{a} \sin \frac{n\pi z}{b}$$

(with $\alpha_{mn} \equiv \pi\sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2}$)

Apply b.c. 6), for $x = 0$:

$$V(0,y,z) = V_0 = \sum_m \sum_n C_{m,n} \sin \frac{m\pi y}{a} \sin \frac{n\pi z}{b}$$

This is a double Fourier series, and can be solved using "Fourier's trick" twice:

Multiply by $\sin \frac{m'\pi y}{a} \sin \frac{n'\pi z}{b}$ and integrate,

$$\int_0^b \int_0^a V_0 \sin \frac{m'\pi y}{a} \sin \frac{n'\pi z}{b}\, dy\, dz = \sum_m \sum_n C_{m,n} \underbrace{\int_0^a \sin \frac{m\pi y}{a} \sin \frac{m'\pi y}{a}\, dy}_{= \frac{a}{2}\delta_{mm'}} \times \underbrace{\int_0^b \sin \frac{n\pi z}{b} \sin \frac{n'\pi z}{b}\, dz}_{= \frac{b}{2}\delta_{nn'}}$$

$\frac{1}{2}$ - period suffices (odd function)

$$= C_{m',n'} \left(\frac{ab}{4}\right)$$

So, assuming V_0 = constant,

$$C_{m,n} = \frac{4V_0}{ab} \underbrace{\int_0^a \sin \frac{m\pi y}{a}\, dy}_{\frac{2a}{m\pi},\ m \text{ odd}} \underbrace{\int_0^b \sin \frac{n\pi z}{b}\, dz}_{\frac{2b}{n\pi},\ n \text{ odd}} = \frac{16 V_0}{\pi^2 mn} \quad \text{if } m,n \text{ both odd}$$

$$\boxed{V(x,y,z) = \frac{16 V_0}{\pi^2} \sum_{m\ \text{odd}} \sum_{n\ \text{odd}} \frac{1}{mn} e^{-\pi\left(\sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2}\right)x} \sin \frac{m\pi y}{a} \sin \frac{n\pi z}{b}}$$

Can do many similar problems -- various sums over sin, cos, sinh, cosh, exp.