Work and Energy in dielectrics (if linear) (~~& isotropic~~)

As our starting point, use the basic expression for the change in energy when charge $\delta\rho_f$ is added to an existing charge distribution:

$$\delta U = \int_\mathcal{V} \delta\rho_f(\vec{r})\, V(\vec{r})\, d\tau$$

Now repeat the derivation of U_{field}, allowing for the presence of a linear dielectric:

We know that

i) $\vec{\nabla}\cdot\vec{D} = \rho_f \Rightarrow \delta\rho_f = \vec{\nabla}\cdot\delta\vec{D}$

ii) Chain rule; $\vec{\nabla}\cdot V\vec{A} = V\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}V$

iii) In a linear isotropic medium $\vec{D} = \varepsilon\vec{E}$, and

$$\vec{E}\cdot\delta\vec{D} = \tfrac{1}{2}\delta(\vec{E}\cdot\vec{D})$$

So, $\delta U = \int_\mathcal{V}(\vec{\nabla}\cdot\delta\vec{D})\, V(\vec{r})\, d\tau$

$$= \int_\mathcal{V}\vec{\nabla}\cdot(V\,\delta\vec{D})\,d\tau - \int \delta\vec{D}\cdot\vec{\nabla}V\,d\tau$$

(a) (b)

For (a), use div. thm.: $(a) = \oint_S V(\delta\vec{D}\cdot\hat{n})\,da$

If we integrate over all space, $V\to 0$ at least as fast as $1/r$ as $r\to\infty$. $\delta\vec{D}$ falls off at least as fast as $1/r^2$, and $S\propto r^2$. Thus (a) $\to 0$.

For (b), $-\int_\mathcal{V}\delta\vec{D}\cdot\vec{\nabla}V\,d\tau = +\int_\mathcal{V}\delta\vec{D}\cdot\vec{E}\,d\tau$

$$= \tfrac{1}{2}\int_\mathcal{V}\delta(\vec{D}\cdot\vec{E})\,d\tau$$

So finally, $\delta U = \frac{1}{2}\int\delta(\vec{D}\cdot\vec{E})\,d\tau$, and

$$\boxed{U = \frac{1}{2}\int_\mathcal{V}\vec{D}\cdot\vec{E}\,d\tau}$$

Electrostatic energy in materials (linear, ~~isotropic~~ dielectric)

not actually required -- turns out $D_i = \sum_j \varepsilon_{ij}E_j$ is sufficient.

Also, energy density per unit volume $\boxed{u = \frac{1}{2}\vec{D}\cdot\vec{E}}$.

What's the distinction, though, between this and $U_E = \frac{1}{2}\varepsilon_0 \int_V E^2 \, d\tau$ = energy in field alone?

It's that U is the total electrostatic energy, including the energy needed to polarize the dielectric. So apparently the energy stored internally in the medium is

$\Delta U = U - U_E$, which is built up as $\vec{E}$ is first applied.

Ex: Griffiths Ex. 4.9, modified:

Find U, U_E for a sphere of radius R, with uniform free charge density ρ_f and dielectric constant ε_r:

Gauss' law $\Rightarrow$ $\vec{D} = \begin{cases} \frac{\rho_f}{3} r \hat{r}, & r < R \\ \frac{\rho_f}{3}\frac{R^3}{r^2}\hat{r}, & r > R \end{cases}$

So $\vec{E} = \frac{\vec{D}}{\varepsilon} = \begin{cases} \frac{\rho_f}{3\varepsilon_0\varepsilon_r} r \hat{r}, & r < R \\ \frac{\rho_f}{3\varepsilon_0}\frac{R^3}{r^2}\hat{r}, & r > R \end{cases}$ ← Same as Problem (4.20)

and $\vec{P} = \varepsilon_0(\varepsilon_r - 1)\vec{E} = \frac{\rho_f}{3}\left(1 - \frac{1}{\varepsilon_r}\right) r \hat{r}$, $r < R$ (only)

Total energy is

$$U = \frac{1}{2}\int \vec{D}\cdot\vec{E}\, d\tau = \frac{1}{2}\iiint_0^R \left(\frac{\rho_f^2}{9\varepsilon_0\varepsilon_r} r^2\right) r^2 \, dr \, d\Omega$$

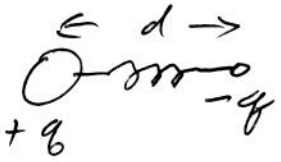
$$+ \frac{1}{2}\iiint_R^\infty \left(\frac{\rho_f^2 R^6}{9\varepsilon_0}\frac{1}{r^4}\right) r^2 \, dr \, d\Omega$$

$$= 2\pi \frac{\rho_f^2}{9\varepsilon_0}\left(\frac{R^5}{5\varepsilon_r} + R^6\frac{1}{R}\right) = \frac{2\pi}{9\varepsilon_0}\rho_f^2 R^5\left(\frac{1}{5\varepsilon_r} + 1\right)$$

The field energy differs only by the $\frac{1}{\varepsilon_r^2}$ in E^2 for $r < R$,

$$U_E = \frac{2\pi}{9\varepsilon_0}\rho_f^2 R^5\left(\frac{1}{5\varepsilon_r^2} + 1\right)$$

↑ try to understand this, using a linear response model with classical oscillators for atoms/molecules



For static balance, need $|F| = qE = kd$,

so $d = \frac{qE}{k}$, or $q = \frac{kd}{E}$

Dipole moment is $p = qd$, and contribution to internal energy is $dU_{int} = \frac{1}{2} k d^2$.

Using $p = \left(\frac{kd}{E}\right) d = \frac{kd^2}{E}$, $dU_{int} = \frac{1}{2} E p$.

Finally, $p = P\, d\tau$ ($d\tau$ = volume of avg oscillator), so

$$U_{int} = \iiint \frac{1}{2} E P \, d\tau, \quad \text{with } P = \frac{\rho_f}{3}\left(1 - \frac{1}{\varepsilon_r}\right) r, \; r < R$$

$$= \frac{1}{2}(4\pi) \int_0^R \frac{\rho_f^2}{9\varepsilon_0 \varepsilon_r}\left(1 - \frac{1}{\varepsilon_r}\right) r^2 (r^2 dr)$$

$$= \frac{2\pi}{9\varepsilon_0} \rho_f^2 \frac{R^5}{5}\left(\frac{1}{\varepsilon_r} - \frac{1}{\varepsilon_r^2}\right)$$

$$= U - U_E \quad \checkmark$$

Note that the values of k and q did not matter — it's really just the linear response of the system that is relevant.

So both U and U_E are "valid," but they contain different information.

A somewhat related question is "what field is actually seen by an individual molecule?" — will answer this very shortly. It does not impact the validity of $\frac{1}{2}\int \vec{E} \cdot \vec{D}\, d\tau$, though.

Ex 2: Energy in a capacitor

For a given V_{ab}, $\begin{cases} E \text{ stays the same as in vacuum} \\ D \text{ increases by } \varepsilon_r \end{cases}$

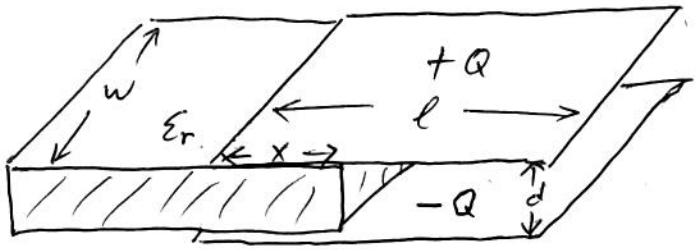
so $U \to \varepsilon_r U_{vacuum}$

But $C \to \varepsilon_r C_{vacuum}$, so

$\boxed{U = \frac{1}{2} C V^2}$ is still correct

Ex 3: (Classic problem) Find the force drawing a dielectric slab into a capacitor. We know that forces due to fringing fields must pull it in, since this leads to a lower-energy configuration, $U = \frac{1}{2} C V^2 \xrightarrow[\text{constant } Q]{} \frac{1}{2}(\varepsilon_r C)(V/\varepsilon_r)^2$.

We will use $\vec{F} = -\vec{\nabla} U$ to find the force without detailed knowledge of the fringing fields. To find the gradient, can either hold V = constant or Q = constant. But the answer must be the same either way, so we will pick Q = constant because it's easier this way.



Assume the plates are large enough that most of the energy is in the region of uniform vertical fields.

$\vec{E} = V/d\,(-\hat{z})$ is uniform and vertical, since $\vec{E}''$ is continuous at the dielectric boundary.

Capacitance $C = C_\ell + C_r$

$$= \varepsilon_r \frac{\varepsilon_0 w x}{d} + \frac{\varepsilon_0 w (\ell - x)}{d}$$

$$C = \frac{\varepsilon_0 w}{d}(\varepsilon_r x + \ell - x)$$

$U = \frac{1}{2} C V^2$ and $C = \frac{Q}{V}$, so

$U = \frac{1}{2} \frac{Q^2}{C}$ (allows calculation of $\vec{\nabla} U$ at fixed Q)

$$\vec{F} = -\frac{\partial U}{\partial x}\hat{x} = +\frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx}\hat{x}$$

$$\vec{F} = \frac{1}{2} V^2 (\varepsilon_r - 1) \frac{\varepsilon_0 W}{d} \hat{x}$$

If we had held V = constant, we would obtain $-\vec{F}$!
Reason: power supply delivers energy $\Delta U = \Delta Q\, V$, which is twice ΔU_{cap} and opposite in sign. So the correct $\vec{F}$ is the same as above, after incorporating this.

Macroscopic, microscopic and molecular fields

The true microscopic field inside a dielectric is wildly inhomogeneous and fluctuating. How do we know that the potential we have been using,

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\vec{P}(\vec{r}\,') \cdot \hat{\mathcal{R}}}{\mathcal{R}^2}\, d\tau',$$

is the correct macroscopic average? Griffiths addresses this in Section 4.2.3 with a variation of the mean value theorem, although one might argue this is something of a pedantic issue.

A more interesting question is, what is the average electric field experienced by an individual molecule in the dielectric (the "molecular field")? You might think that if $\vec{p} = \alpha \vec{E}$ for a single molecule, then $\vec{P} = N\alpha\vec{E}$, but this is <u>not</u> accurate!

To handle this, remove a sphere of radius R from the dielectric and replace with a sum over the individual molecules. Need $R \gg$ molecule spacing, but $\ll$ scale over which $\vec{E}$ varies. Then for a molecule at its center,

$$\vec{E}_m = \vec{E} + \vec{E}_p + \vec{E}'$$

- $\vec{E}$: Field in bulk material
- $\vec{E}_p$: field of surface polarization charge density
- $\vec{E}'$: sum over individual mol's

$\vec{E}_p$ comes from integral of $\sigma_b = \vec{P}\cdot\hat{n}$ over a sphere, and is identical to the problem of the uniformly polarized sphere except for a (−) sign:

$$\vec{E}_p = \frac{1}{3\varepsilon_0}\vec{P}$$

Also, $\vec{E}' = \sum E_i' \to 0$ for substances with randomly oriented molecules (liquids, gases, glass...) or symmetric crystal lattices. (If not, get birefringence, other corrections).

Assuming $\vec{E}' = 0$ (isotropic medium),

$$\boxed{\vec{E}_m = \vec{E} + \vec{E}_p = \vec{E} + \frac{1}{3\varepsilon_0}\vec{P}}$$

The molecular dipole $\vec{p}$ is produced by $\vec{E}_m$,

$$\vec{p} = \alpha \vec{E}_m \qquad \text{(if polarizability is } \alpha\text{)}$$

Using $\vec{P} = n\vec{p}$,

$$\boxed{\frac{\vec{P}}{n\alpha} = \vec{E} + \frac{1}{3\varepsilon_0}\vec{P}} \qquad \left(P \neq n\alpha \vec{E}\right)$$

$$= \frac{P}{\varepsilon_0 \chi_e}$$

So $$\boxed{\frac{1}{\varepsilon_0 \chi_e} = \frac{1}{n\alpha} - \frac{1}{3\varepsilon_0}}$$ ← "local field" correction, basically, because molecule should not be included in calculation of V.

Can be rearrange using $\vec{P} = \varepsilon_0 (\varepsilon_r - 1)\vec{E}$ to

$$\boxed{\alpha = \frac{3\varepsilon_0}{n} \frac{\varepsilon_r - 1}{\varepsilon_r + 2}}$$ Clausius - Mosotti equation