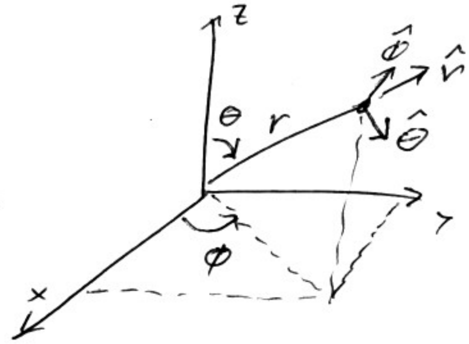
Spherical and Cylindrical coordinates

We will increasingly be working with spherical & cylindrical configurations as time goes on, so let's fill in the missing pieces from ~~Chapter~~ Section 1.4:

Spherical Coordinates:



θ: polar angle, $0 \to \pi$

Φ: azimuthal, $0 \to 2\pi$

$$x = r \sin\theta \cos\Phi$$

$$y = r \sin\theta \sin\Phi$$

$$z = r \cos\theta$$

$\hat{r}, \hat{\theta}, \hat{\Phi}$ vary with position but are always orthogonal.

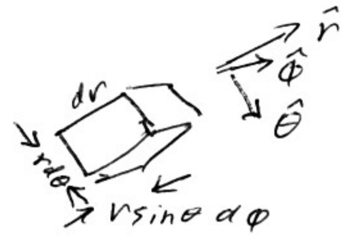
In terms of Cartesian coords,

$$\hat{r} = \sin\theta \cos\Phi \, \hat{x} + \sin\theta \sin\phi \, \hat{y} + \cos\theta \, \hat{z}$$

$$\hat{\theta} = \cos\theta \cos\Phi \, \hat{x} + \cos\theta \sin\Phi \, \hat{y} - \sin\theta \, \hat{z}$$

$$\hat{\Phi} = -\sin\phi \, \hat{x} + \cos\phi \, \hat{y}$$

Differentials:



We already examined the differential box at right, from which:

1) $d\tau = r^2 \sin\theta \, d\theta \, d\phi \, dr$

$\quad = dl_r \, dl_\theta \, dl_\phi$

2) $d\Omega = \sin\theta \, d\theta \, d\phi$ (steradians)

3) $\vec{dl} = dr \, \hat{r} + r d\theta \, \hat{\theta} + r \sin\theta \, d\phi \, \hat{\Phi}$

But $\vec{da}$ is specific to each situation. For the surface of a sphere, it's easy: $\vec{da}_{sphere} = dl_\theta \, dl_\phi \, \hat{r}$

$$= r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$

Vector derivatives:

In principle, just use chain rule for partial derivatives. Very messy, though.
For example, $\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta}$! (Can easily verify using expression for $\hat{r}$ in terms of Cartesian coords).

A general approach is shown in App. A of Griffiths; resulting formulas are on inside front cover.

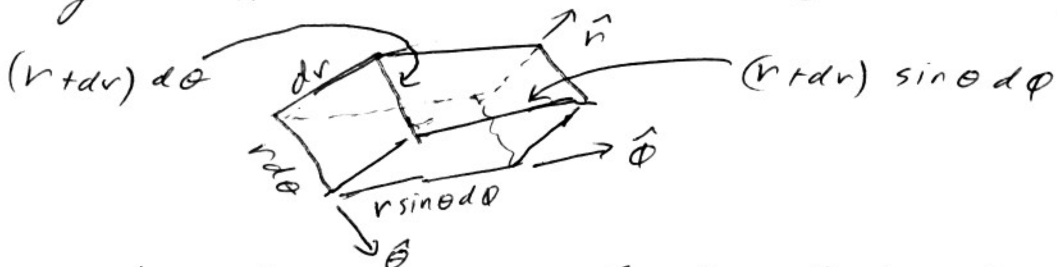
Let's look at a version of this specific to spherical coordinates, for the case of the divergence.

Define divergence via $\int \vec{\nabla}\cdot\vec{A}\, d\tau = \oint_s \vec{A}\cdot d\vec{a}$ $\left(= \oint \vec{A}\cdot\hat{n}\, da\right)$

Taking a volume so small that $\vec{\nabla}\cdot\vec{A} \simeq$ constant,

$$\vec{\nabla}\cdot\vec{A} \equiv \lim_{\Delta\tau\to 0} \frac{\oint_s \vec{A}\cdot d\vec{a}}{\Delta\tau}$$

Returning to the differential box (draw it bigger),



For the two faces $\perp$ to $\hat{r}$ (top & bottom),

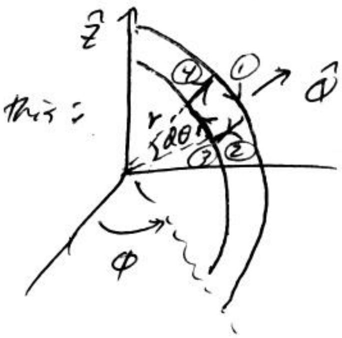
$$\oint_s \vec{A}\cdot d\vec{a} = A_r(r+dr,\theta,\phi)\,\big((r+dr)\,d\theta\big)\big((r+dr)\sin\theta\, d\phi\big)$$
$$- A_r(r,\theta,\phi)\,(r\,d\theta)(r\sin\theta\, d\phi)$$
$$= \sin\theta\, d\theta\, d\phi\,\left((r+dr)^2 A_r(r+dr) - r^2 A_r(r)\right)$$
$$= \sin\theta\, d\theta\, d\phi\left(\frac{\partial}{\partial r}\left(r^2 A_r\right)\right) dr$$

So $$\vec{\nabla}\cdot\vec{A}\Big|_{\text{for } \hat{r} \text{ faces}} = \frac{\oint \vec{A}\cdot d\vec{a}}{r^2\sin\theta\, d\theta\, d\phi\, dr}$$
$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right)$$

Adding on the $\hat{\theta}$ and $\hat{\phi}$ faces we get

$$\boxed{\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}}$$

Curl can be done similarly via Stokes' theorem:

For $\hat{\phi}$ component, use this: 

$\hat{\phi}$ ($\perp$ to plane of loop)

$$\vec{\nabla}\times\vec{A}\Big|_u = \lim_{\Delta a \to 0} \frac{\oint \vec{A}\cdot d\vec{\ell}}{\Delta a},$$

$\hat{u}$ $\perp$ to surface.

~~Skipping~~ For 1 and 3,

$$\int_{1+3} \vec{A}\cdot d\vec{\ell} = A_\theta(r+dr, \theta, \phi)(r+dr)d\theta - A_\theta(r, \theta, \phi)\, r\, d\theta$$

$$= \frac{\partial}{\partial r}(r A_\theta)\, d\theta\, dr$$

sim. for 2 and 4, giving

$$\vec{\nabla}\times\vec{A}\Big|_\phi = \frac{\oint \vec{A}\cdot d\vec{\ell}}{r\, d\theta\, dr} = \left(\frac{1}{r}\frac{\partial}{\partial r}(r A_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)\hat{\phi}$$

Likewise for $\hat{r}$, $\hat{\theta}$. See formulas in text.

Gradient is a little easier. Use geometric interpretation - points in direction in which Φ increases fastest, magnitude = rate of increase.

$$\frac{d\psi}{d\ell}\Big|_{\text{in direction } \hat{\ell}} = \vec{\nabla}\psi\cdot\hat{\ell} \quad \left(\text{same as } d\psi = \vec{\nabla}\psi\cdot d\vec{\ell}\right)$$

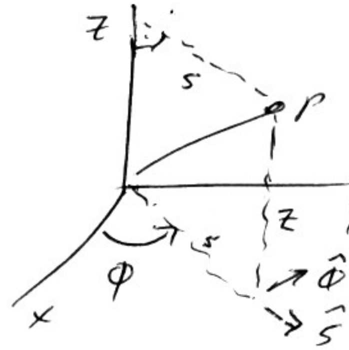
Now, $d\psi = \frac{\partial \psi}{\partial r}dr + \frac{\partial \psi}{\partial \theta}d\theta + \frac{\partial \psi}{\partial \phi}d\phi$,

and $d\vec{\ell} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\, d\phi\,\hat{\phi}$

So. $d\psi = \left(\frac{\partial \psi}{\partial r}\hat{r} + \frac{\partial \psi}{\partial \theta}\frac{1}{r}\hat{\theta} + \frac{\partial \psi}{\partial \phi}\frac{1}{r\sin\theta}\hat{\phi}\right)\cdot d\vec{\ell}$

or $$\frac{d\psi}{d\ell} = \underbrace{\left(\frac{\partial \psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \phi}\hat{\phi}\right)}_{\vec{\nabla}\psi}\cdot\hat{\ell}$$

Cylindrical coordinates



s = distance from z axis ($0 \to \infty$)

ϕ = angle from x axis ($0 \to 2\pi$)

$z = z$ ($-\infty$ to $+\infty$)

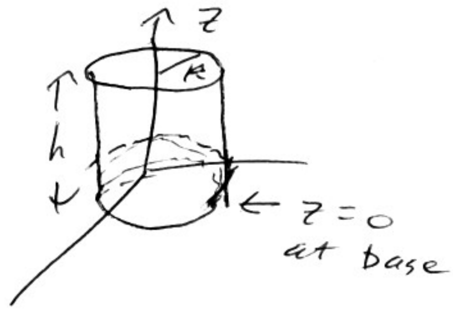
$$d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$$

$$d\tau = s\,ds\,d\phi\,dz$$

$$\hat{s} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$
$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$
$$\hat{z} = \hat{z}$$

For vector derivatives see book cover.

Ex: Volume of a cylinder



$$V = \int_{cyl} d\tau = \int_0^R \int_0^{2\pi} \int_0^h s\,ds\,d\phi\,dz$$

$$= \int_0^R s\,ds \int_0^{2\pi} d\phi \int_0^h dz = \frac{R^2}{2}(2\pi)(h)$$

$$= \pi R^2 h$$

Note: in most texts, s is called r.

The Dirac delta function (sec. 1.5)

Paradox?

Consider Gauss' law, $\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{encl}$

By the divergence thm, $\int_V \vec{\nabla}\cdot\vec{E}\, d\tau = \frac{1}{\varepsilon_0} Q_{encl}$

For a sphere around a point charge q, at the origin,

$$\int_{V \text{ of sphere}} (\vec{\nabla}\cdot\vec{E})\, d\tau = \frac{q}{\varepsilon_0} \quad (\text{for } \underline{\text{any}} \text{ radius } r)$$

But we know $\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$

and so $\vec{\nabla}\cdot\vec{E} = \text{"} \frac{1}{r^2}\left(\frac{\partial}{\partial r}\frac{r^2}{r^2}\right) = 0$,

so $0 = \frac{q}{\varepsilon_0}$!

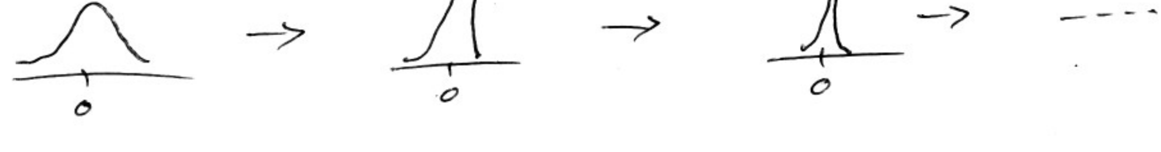
Solution: At the origin, $\vec{\nabla}\cdot\vec{E} \to \infty$, so it's not a well-behaved function. Apparently any volume integral that includes the origin will have the value q/ε_0. $\vec{\nabla}\cdot\vec{E}$ for $\frac{\hat{r}}{r^2}$ is an example of a <u>delta</u> "<u>function</u>".

1D delta function

Defined by: $\delta(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ \infty & \text{for } x = 0 \end{cases}$

and $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$

So it's a singularity with area $= 1$ -- only a useful concept inside an integral. In mathematics it's a "generalized function" or type of "functional". It's the limit of a series of functions ~~of~~ area 1 that get steeper and narrower:



Because $\delta(x) = 0$ except at $x=0$,

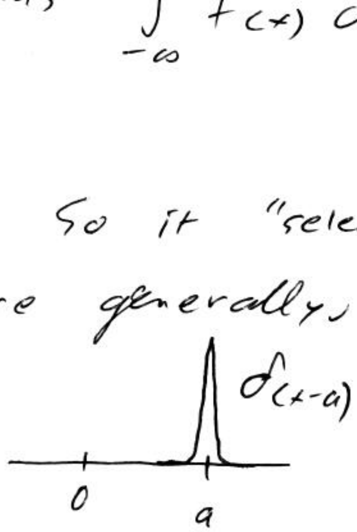
$$f(x)\,\delta(x) = f(0)\,\delta(x) \quad \text{for any ordinary function } f(x).$$

Thus $\displaystyle\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)\int_{-\infty}^{\infty}\delta(x)\,dx$ (actually, for any limits that include $x=0$)

$$= f(0)$$

So it "selects" the point $x=0$ for evaluation.

More generally, shift the origin:



$$\boxed{\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)}$$

So $\delta(x-a)$ selects the value of the integrand at $x=a$.

For example, $\displaystyle\int_{-\infty}^{\infty}(x^2+1)\,\delta(x+1)\,dx = (-1)^2+1 = 2$

$$= \int_{-2}^{0}(x^2+1)\,\delta(x+1)\,dx$$

But $\displaystyle\int_{0}^{\infty} = \int_{0}^{2} = 0$ because $x=-1$ is outside range.

There are several peculiar properties of δ-fcns.

Ex: what does $\delta(kx)$ do?

Consider $\displaystyle\int_{-\infty}^{\infty} f(x)\,\delta(kx)\,dx$.

let $u = kx$, so $x = \frac{u}{k}$, $dx = \frac{1}{k}du$.

As x ranges $-\infty \to \infty$, u ranges $\begin{cases} -\infty \to \infty & \text{if } k>0 \\ \infty \to -\infty & \text{if } k<0 \end{cases}$

So $\displaystyle\int_{-\infty}^{\infty} f(x)\,\delta(kx)\,dx = \pm\int_{-\infty}^{\infty} f\!\left(\frac{u}{k}\right)\delta(u)\,\frac{1}{k}\,du$

$$= \frac{1}{|k|}f(0)$$

Thus $\delta(kx)$ has the same effect as $\frac{1}{|k|}\delta(x)$, so we can ~~identify~~ equate them:

$$\boxed{\delta(kx) = \frac{1}{|k|}\delta(x)}$$

There are some others, too --- see problem 1.46.

3D delta function

This is defined in the obvious way in Cartesian coordinates:

$$\delta^3(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z) \quad (\text{zero except at } \vec{r}=0)$$

So $\displaystyle\int_{\mathcal{V}} \delta^3(\vec{r})\, d\tau = 1$
all space

and $\displaystyle\int_{\mathcal{V}} f(\vec{r})\,\delta^3(\vec{r}-\vec{a})\, d\tau = f(\vec{a})$
all space

For a finite volume,

$$\int_{\mathcal{V}} \delta^3(\vec{r}-\vec{a})\, d\tau = \begin{cases} 1 & \text{if } \vec{a} \text{ in } \mathcal{V} \\ 0 & \text{if not.} \end{cases}$$

Divergence of $\frac{\hat{r}}{r^2}$:

Going back to Gauss' law + div. theorem, need

$$\int_{\mathcal{V}} \vec{\nabla}\cdot\vec{E}\, d\tau = \frac{q}{\varepsilon_0}, \quad \vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$$

for any $\mathcal{V}$ enclosing origin. This is exactly like a delta function..

If we write $\vec{\nabla}\cdot\vec{E} = K\,\delta^3(\vec{r})$, then for $\mathcal{V}$ = sphere around origin, (or any other shape)

$$\int \vec{\nabla}\cdot\vec{E}\, d\tau = K = \frac{q}{\varepsilon_0}$$

So $\vec{\nabla}\cdot\frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2} = \frac{q}{\varepsilon_0}\delta^3(\vec{r})$, or

$$\boxed{\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = 4\pi\delta^3(\vec{r})}$$

Generalize to a charge at $\vec{r}\,'$:

$$\boxed{\vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 4\pi\delta^3(\vec{\mathscr{r}})} \quad , \vec{\mathscr{r}} \equiv \vec{r}-\vec{r}\,'$$

Also, since $\vec{\nabla}\frac{1}{\mathscr{r}} = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$ (problem 1.13),

$$\nabla^2\frac{1}{\mathscr{r}} = \vec{\nabla}\cdot\left(-\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$$

$$\boxed{\nabla^2\frac{1}{\mathscr{r}} = -4\pi\delta^3(\vec{\mathscr{r}})}$$ (will use this later)

Ex: Easy derivation of differential form of Gauss' law

Start with Coulomb's law for a continuous charge distribution,

$$\vec{E} = \left(\int_{\text{all space}} \rho(\vec{r}\,')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'\right)\left(\frac{1}{4\pi\varepsilon_0}\right)$$

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{4\pi\varepsilon_0}\int\rho(\vec{r}\,')\left(\vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau'$$

$\vec{\nabla}$ operates on $\vec{r}$, not $\vec{r}\,'$

$$= \frac{1}{4\pi\varepsilon_0}\int\rho(\vec{r}\,')\left(4\pi\delta^3(\vec{\mathscr{r}})\right)d\tau'$$

$$\boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho(\vec{r})}{\varepsilon_0}}$$ easy!

Ex: $\boxed{\rho(\vec{r}) \text{ for a point charge at } \vec{r}_i = q\,\delta^{(3)}(\vec{r}-\vec{r}_i)}$

For a point charge at the origin,

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int \rho(\vec{r}\,') \frac{\hat{\imath}}{\imath^2} \, d\tau'$$

$$\rho(\vec{r}\,') = q \, \delta^3(\vec{r}\,')$$

So $$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int q \underbrace{\delta^3(\vec{r}\,')}_{\text{picks } \vec{r}\,'=0} \frac{\widehat{\vec{r}-\vec{r}\,'}}{|\vec{r}-\vec{r}\,'|^2} \, d\tau'$$

$$= \frac{q}{4\pi\varepsilon_0} \frac{\hat{r}}{r^2} \quad \checkmark$$

For charge at $\vec{r}_i$,

$$E = \frac{1}{4\pi\varepsilon_0} \int q \, \delta^3(\vec{r}\,' - \vec{r}_i) \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \, d\tau'$$

$$= \frac{q}{4\pi\varepsilon_0} \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|^3} \quad \checkmark$$

So if we want to, we can treat point charges along with volume charge density.