Physics 3201 - E&M I

You have already encountered most of the basic concepts in Physics 1402/1502/1602 or similar:

static charges → Coulomb's law → $\vec{E}$ → Gauss' law
moving charges → Biot - Savart law → $\vec{B}$ → Ampère's law
changing fields → Faraday's law & massive confusion (Maxwell's eqn's, waves)

Here we will be more systematic and detailed, using vector calculus to obtain a full description. The paradigm is also a little different —

Charges → fields → forces,
with rules given by Maxwell's equations & Lorentz force law.

So this will mostly be a course about EM <u>fields</u>.

But the foundation is <u>electric</u> <u>charge</u>. On the basis of empirical evidence,

1. It comes in two cancelling varieties, (+) and (−).
2. Charge is conserved, locally and globally.
3. Charge is quantized, $e = 1.602\ldots \times 10^{-19}$ C.

In this first half of the 3201/3202 sequence we will focus mainly on <u>statics</u> — stationary charges or steady currents. However, towards the end we will begin to explore the overlap with basic electrodynamics.

We will cover Chs. 1−6 of Griffiths, with a little supplementation in spots. Initially we will skip back and forth between Chs. 1 and 2. This is done so we don't have to deal with all of the mathematics in one huge chunk, without seeing anything of its uses for describing physics.

First, a word about Units

We will use SI units. However, relativistic E&M is often still done in cgs/gaussian units.

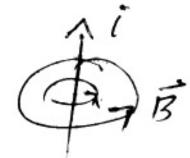
Two parameters define the EM unit system:

1. In Coulomb's law for a pair of charges,

$$\vec{F}_1 = \alpha \frac{q_1 q_2}{r_{12}^2} \hat{r}_{12} = \alpha \frac{q_1 q_2}{\mathscr{r}^2} \hat{\mathscr{r}}$$

2. In magnetic induction, as we will see in a few weeks, the $\vec{B}$ field around a wire is given by

$$|\vec{B}| = \beta \frac{I}{d}$$



Here α, β determine the relation between mechanical and EM units. We are free to choose their values.

system	α	β	mech. units
SI / rationalized	$\frac{1}{4\pi\varepsilon_0}$	$2\frac{\mu_0}{4\pi}$	SI
Heaviside - Lorentz	$\frac{1}{4\pi}$	$\frac{2}{4\pi c}$	cgs
Gaussian	1	$\frac{2}{c}$	cgs (usual cgs choice)
electromagnetic (emu)	c^2	2	cgs
electrostatic (esu)	1	$\frac{2}{c^2}$	cgs

So the dimensions depend on the unit system!

(In Gaussian units, capacitance in cm, resistance in $\frac{s}{cm}$)

For the SI system,

$$\frac{1}{4\pi\varepsilon_0} = 10^{-7} c^2 \left(\text{in } F^{-1} s^2 m^{-1}\right), \text{ or } \varepsilon_0 = 8.85\ldots \times 10^{-12} \frac{F}{m}$$

$$\frac{2\mu_0}{4\pi} = 2 \times 10^{-7} \left(\text{in } N A^{-2}\right), \text{ or } \mu_0 = 4\pi \times 10^{-7} \frac{N}{A^2}$$

$$\frac{1}{\mu_0 \varepsilon_0} = c^2$$

For SI $\Leftrightarrow$ gaussian equivalencies see App. C of Griffiths.

<u>Vectors</u> (Griffiths Sec. 1.1) in Cartesian coord's

Archetype is <u>displacement</u> <u>vector</u> from A to B:

has magnitude, direction

Special case is the <u>position</u> <u>vector</u> $\vec{r}$ from the origin to a point x, y, z,

$\vec{r} = (x, y, z)$ (row vector)

or $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (column vector)

or (r_1, r_2, r_3) $i = 1, 2, 3$ gives component r_i

or $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

Magnitude $r = \sqrt{x^2 + y^2 + z^2}$

and $\hat{r} = \frac{\vec{r}}{r}$ is radial unit vector (points outward).

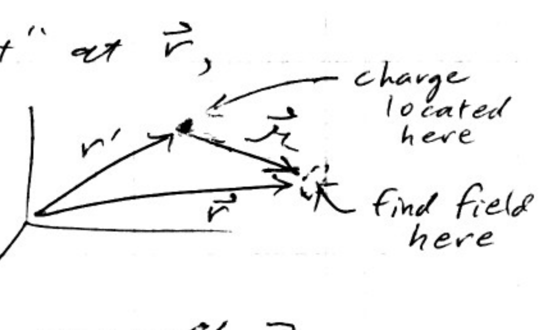
For <u>infinitesimal displacements</u> Griffiths likes to use

$$d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}.$$

Finally, for <u>separation</u> <u>vector</u> between "source point" at $\vec{r}'$ and "field point" at $\vec{r}$,

$\vec{\mathscr{r}} \equiv \vec{r} - \vec{r}'$

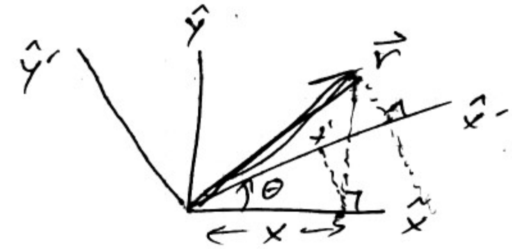
$\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}} = \frac{\vec{r} - \vec{r}'}{|r - r'|}$



What defines a vector more generally?

1. piece of x-axis of coord system? No
2. (monkey, echinoderm, phenomenology) NO!

Key is behavior under coordinate system rotation



$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$
$$z' = z,$$

or $$r_i' = \sum_{j=1}^{3} a_{ij}\, r_j\,, \qquad (1)$$

$$a_{ij} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(Here the summation is equiv. to
$$\begin{pmatrix} r_1' \\ r_2' \\ r_3' \end{pmatrix} = \begin{pmatrix} \quad & \quad & \quad \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}. \;)$$

A <u>general</u> rotation needs three angles. Still, always a linear transformation, like (1), to keep straight lines straight.

(Also, to preserve lengths, a_{ij}'s must be orthogonal,
$$a^T = a^{-1}, \text{ or } \sum_i a_{ij} a_{ik} = \delta_{jk} \quad (1 \text{ iff. } j=k) \;)$$

I. A (3-) <u>vector</u> $\vec{A}$ is a 3-component object that transforms under rotations according to
$$A_i' = \sum_j a_{ij} A_j,$$
where a_{ij} is the same orthogonal matrix that appears in the rotation of the position vector.

II. A <u>scalar</u> is a one-component object that is unchanged under rotations.

III. (more later) A 2nd-rank Cartesian tensor T is a 9-component object that transforms under rotations according to

$$T'_{ij} = \sum_{k,l} a_{ik}\, a_{jl}\, T_{kl}$$

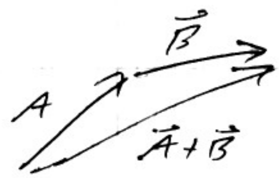
3rd-rank tensor has $3 \times 3 \times 3 = 27$ components,

$$T'_{ijk} = \sum_{l,m,n} a_{il}\, a_{jm}\, a_{kn}\, T_{lmn}$$

(vector = first-rank tensor).

Basic vector operations:

A. Addition – Add components



$$\vec{A} + \vec{B} = (A_x + B_y)\,\hat{x} + (A_y + B_y)\,\hat{y} + (A_z + B_z)\,\hat{z}$$

Commutative: $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

associative: $(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$

To subtract just add $-\vec{B}$:

$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

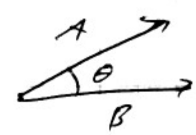
B. Multiplication by scalar -- just increase magnitude by a (but if a negative, reverse direction)

$$a\vec{A} = aA_x\,\hat{x} + aA_y\,\hat{y} + aA_z\,\hat{z}$$

distributive: $a(\vec{A} + \vec{B}) = a\vec{A} + a\vec{B}$

and commutative

C. Scalar "dot" product



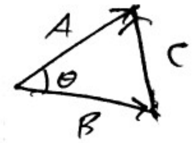
$\vec{A} \cdot \vec{B} = AB\cos\theta$

$= A_x B_x + A_y B_y + A_z B_z$

Commutative and distributive, $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$

Note $\vec{A} \cdot \vec{A} = A^2$, $\hat{x} \cdot \hat{x} = 1$, $\hat{x} \cdot \hat{y} = 0$

Quick proof of law of cosines



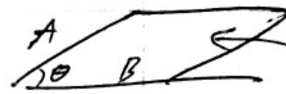
$$\vec{C} = \vec{A} - \vec{B}$$

$$C^2 = \vec{C}\cdot\vec{C} = (\vec{A}-\vec{B})\cdot(A-\vec{B}) = A^2 + B^2 - 2\,\vec{A}\cdot\vec{B}$$

$$C^2 = A^2 + B^2 - 2AB\cos\theta$$

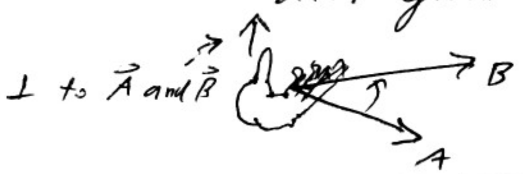
D. Vector "cross" product

$$\vec{A}\times\vec{B} = AB\sin\theta\ \hat{n},$$



area of paralellogram

$\hat{n}$ = unit vector $\perp$ to plane of $\vec{A}$ and $\vec{B}$;
dir. given by rt. hand rule



(curl fingers in shortest way)

So $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$ (anticommutes)

$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$ (distributive)

$\vec{A}\times\vec{A} = 0$ (for any vector, since $\theta = 0$, but not obvious for $\vec{\nabla}$)

Note $\hat{x}\times\hat{y} = \hat{z}$, $\hat{y}\times\hat{z} = \hat{x}$, $\hat{z}\times\hat{x} = \hat{y}$ ("cyclic" permutations)

By components,

$$\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{x} + (A_zB_x - A_xB_z)\hat{y} + (A_xB_y - A_yB_x)\hat{z}$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$(\vec{A}\times\vec{B})_i = \sum_{j,k} \epsilon^{ijk} A_j B_k\,,$$

$$\epsilon^{123} = \epsilon^{231} = \epsilon^{312} = 1 \qquad \text{all others} = 0$$
$$\epsilon^{213} = \epsilon^{132} = \epsilon^{321} = -1$$

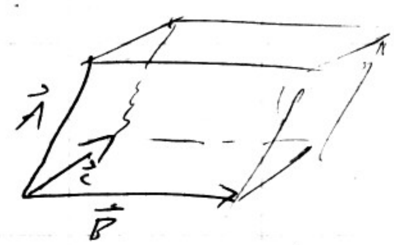
"completely antisymmetric unit tensor"

Triple products

1) Scalar triple product

$\vec{A}\cdot(\vec{B}\times\vec{C})$ (gives volume of parallelepiped with sides $\vec{A}$, $\vec{B}$, $\vec{C}$)

$\vec{B}\times\vec{C}$ = area of base, $\hat{n}$ is $\perp$ to base



Cyclic permutations are equal:

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

Can write as

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

2) Vector triple product

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$ ("BAC-CAB rule")

Not associative – $\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}$

Can be used to simplify higher vector products to have no more than one cross product in any term.