Capacitors and Inductors -- ac Circuits

Capacitors: store electric field

Inductors: store magnetic field

Basic equation of a capacitor is

$$Q = CV$$

C: constant capacitance, in Farads (F) (most often pF or μF.)

Using $I = \frac{dQ}{dt}$, we can write eqn's for behavior of capacitive circuits.

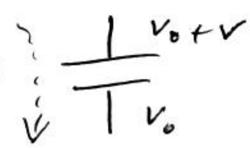
For an inductor,

$$V = L\frac{dI}{dt}$$

So one depends on $\frac{dI}{dt} = \frac{d^2Q}{dt^2}$, the other on $Q = \int I\,dt$.

Signs: for C, $-$ if moving from $+Q$ to $-Q$ ...

$\Delta V = \frac{-Q}{C}$ (−)



(much like resistor)

for L, $-L\frac{dI}{dt}$ in direction of I.

Types:

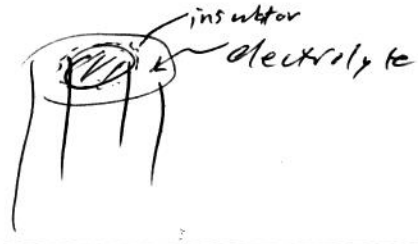
1) Mylar --

general purpose

just roll up!

2) ceramic -- dielectric is a ceramic (good at rf)

3) electrolytic



(+), (−) vital!

- leaky, but high C for size.

4) Metallized plastics (polyester, etc.)

Biggest trick -- reading values!

Usually in either pF or μF. (If in doubt, use a DMM with capacitance ranges)

M means $\pm 20\%$

Voltage rating may or may not be stamped.

use common sense

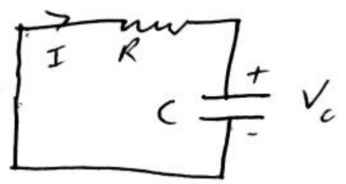
Sometimes coded like precision resistors; exponential labeling —
575 means 57×10^5 pF, or 5.7 μF

Analysis of RC circuits:

Three approaches

1) Direct solution (easiest for impulse response).
2) Response to sinusoidal excitation --- Impedance
3) (later?) Laplace xforms (s-space)

Basic RC circuit (without driving -- transient sol'n)



Assume $V_c = V_0$ at $t=0$

loop theorem:

$$-IR - V_c = 0$$

$$V_c = \frac{Q}{C} = -IR$$

$$\frac{dQ}{dt} = -RC\frac{dI}{dt}$$

or $\dfrac{dI}{dt} = -\dfrac{1}{RC} I$

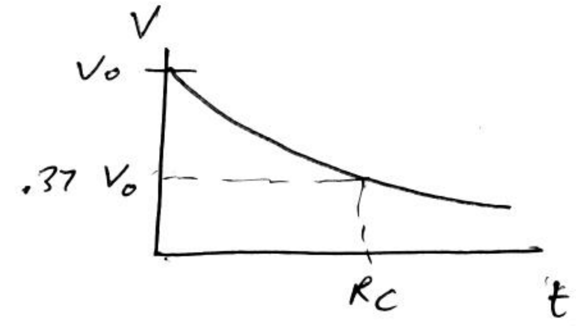
Sol'n: $I = I_0 \, e^{-t/RC}$

or since $I = C\dfrac{dV_c}{dt}$, or ~~better~~ easier, $V_c = IR$,

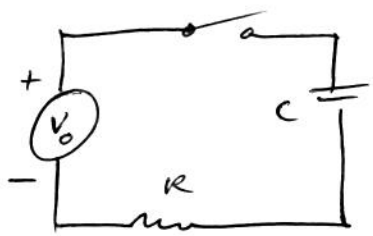
$$\boxed{V = V_0 \, e^{-t/RC}}$$

where $V_0 = I_0 R$ at $t=0$

(will be zero unless circuit looked different before $t=0$.)



This is a discharge cycle; charging is quite analogous:



$$V_0 - V_c - IR = 0$$

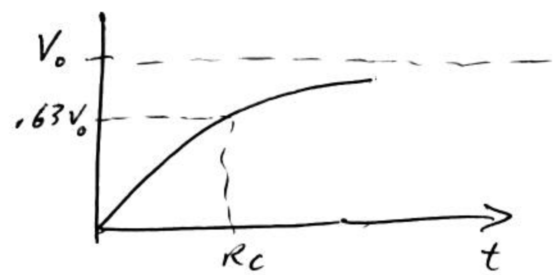
$$V_0 - \frac{Q}{C} - IR = 0$$

$$-\frac{1}{RC} I = \frac{dI}{dt} \quad \text{again.}$$

But now $V_c = V_0 - IR$

So $V_c = V_0 - I_0 R e^{-t/RC}$

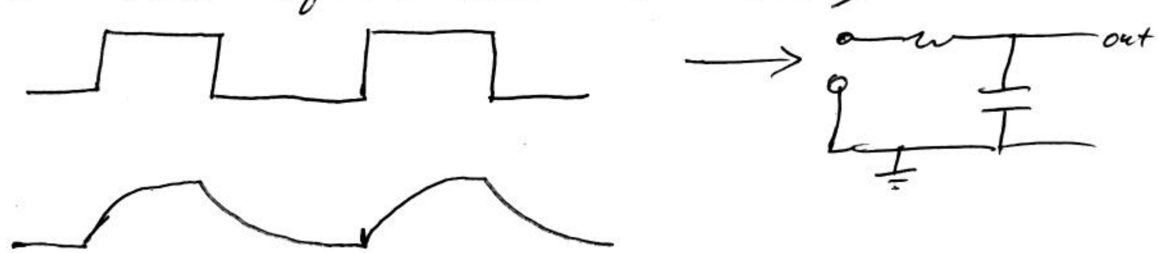
If $V_c = 0$ at $t = 0$, $V_c = V_0 (1 - e^{-t/RC})$



just upside down.

In general, relaxes to the state where $I = 0$ with time constant RC.

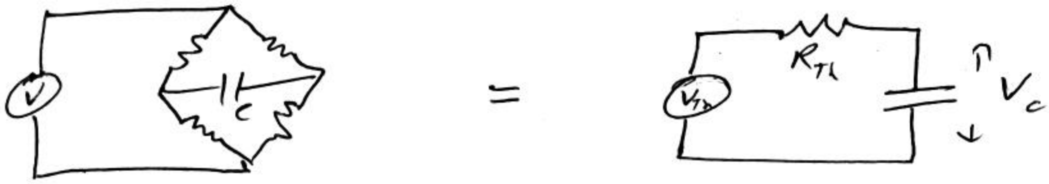
For a slow square wave we see,



For a fast square wave, $T \ll RC$,

(nearly triangular)

For more complex RC's can sometimes use Thevenin equivalent; makes life a lot easier:



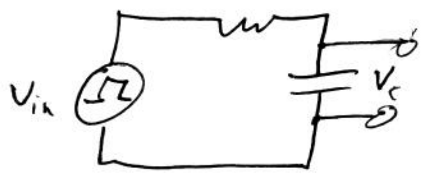
$V_c = ?$

(Actually, we'll see theorem even works for L's & C's in the part we simplify out!)

So far we have a slow-down circuit.

<u>Alternative</u> <u>interpretation</u>:

Can also use it as an <u>integrator</u>:



$$V_{in} - IR - V_c = 0$$

write in terms of V_c this time —

$$I = C\frac{dV_c}{dt} = \frac{V_{in} - V_c}{R}$$

If we make RC <u>large</u>, then $V_c << V_{in}$ and
for small times

$$\frac{dV_c}{dt} \simeq \frac{V_{in}}{RC}$$

and $\boxed{V_c \simeq \frac{1}{RC}\int V_{in}}$ $(+\ V_0)$, if $T << RC$.

This is especially effective with op-amps, where we eliminate the unwanted term altogether!

For square wave, the error is the curvature:

exact

approx. (1st term in Taylor exp'n is what's wanted.)

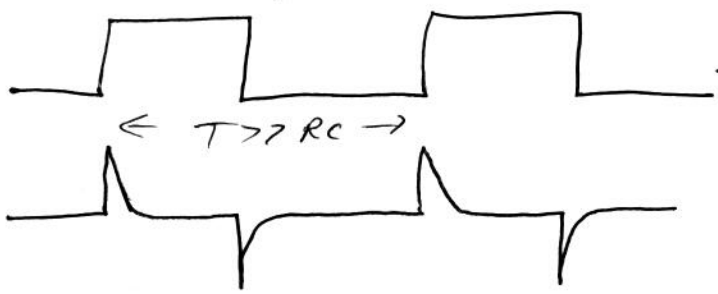
If we look at V_R instead we can form a high-pass or differentiator ---

$$I = \frac{V}{R} = C\frac{dV_c}{dt}$$

$$V_{in} - V_c - V = 0 \quad \Longrightarrow \quad \frac{V}{R} = C\frac{d}{dt}\left(V_{in} - V\right)$$

For small RC we can get $V \gg RC\frac{dV}{dt}$, and

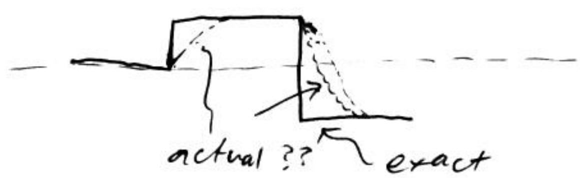
$$\boxed{V \simeq RC\frac{dV_{in}}{dt}}$$



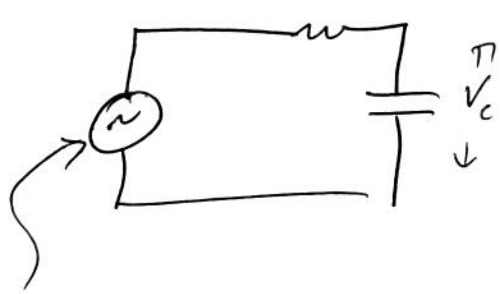
← in (infinite $\frac{dV}{dt}$)

← out

← in (triangle wave)



Response to sinusoidal ~~excitation~~:



(general, using Fourier's theorem)

We want the steady-state solution for V_c.

$V_{in} = V_0 \cos\omega t$

V_0 = amplitude

$V_{rms} = \frac{V_0}{\sqrt{2}}$ = root-mean-square value. $= \frac{V_{p-p}}{2\sqrt{2}}$

Reason: in a resistor $P = I^2 R$ or V^2/R

$$\langle P \rangle_{cycle} = \frac{V_0^2}{R}\langle \cos^2 \omega t \rangle = \frac{1}{2}\frac{V_0^2}{R}$$

$$= \frac{V_{rms}^2}{R}, \quad V_{rms} = \sqrt{\langle (V(t))^2 \rangle_{cycle}}$$

root mean square

So "117 V" (or 120) refers to rms value:

$$\left\{ \begin{array}{l} V_0 = \text{amplitude} \\ 2V_0 = P-P \\ \dfrac{V_0}{\sqrt{2}} = \text{rms value} \end{array} \right]$$ For a sine wave; different for other shapes.

Return to our driven circuit now:

$$V_0 \cos \omega t - IR - V_c = 0$$

with $I = C\dfrac{dV_c}{dt}$,

$$\boxed{V_0 \cos \omega t - RC\frac{dV_c}{dt} - V_c = 0}$$

Optional — Review of Complex Numbers

The idea is to make a guess in complex form,

normally not written out

$$\hat{V}_c = \hat{V}_{c0} e^{i\hat{\omega} t} \quad \text{where}$$

complex amplitude $\hat{V}_{c0} = Ae^{i\phi}$ gives amplitude & phase

$\hat{\omega} = \omega' + i\gamma$ gives freq. & damping

not needed here!

and where the physical solution is $V_c = Re\left(\hat{V}_c\right)$

Proceed in detail: re-write as

$$Re\left(V_0 e^{i\omega t}\right) - RC\frac{d\left(Re\, \hat{V}_c\right)}{dt} - Re\, \hat{V}_c = 0$$

Can easily show $\dfrac{d}{dt}\left(Re\, e^{i\hat{\omega} t}\right) = Re\left(\dfrac{d}{dt} e^{i\hat{\omega} t}\right)$

So $Re\left(V_0 e^{i\omega t} - RC\dfrac{d\hat{V}_c}{dt} - \hat{V}_c\right) = 0$

if $\hat{\omega} = \omega$, for driven response,

$$Re\Big[\underbrace{\left(V_0 - i\omega RC\hat{V}_{c0} - \hat{V}_{c0}\right)}_{\text{must be } 0} e^{i\omega t}\Big] = 0$$

So $\hat{V}_{co} = V_0 \dfrac{1}{1+i\omega RC}$, $V_c = Re\left(\hat{V}_{co}\, e^{i\omega t}\right)$

Looks like a voltage divider if we say that a capacitor has an <u>impedance</u> $Z_c = \dfrac{1}{i\omega C}$:

$$\hat{V}_{co} = V_0 \frac{Z_c}{R + Z_c}$$

To get phase explicitly write $|\hat{V}_{co}|\, e^{i\phi}$

$$\hat{V}_{co} = V_0 \frac{1 - i\omega RC}{1 + (\omega RC)^2} \quad \text{gives } \phi = \tan^{-1}(-\omega RC)$$

$$|\hat{V}_{co}| = \sqrt{V_{co} V_{co}^*} = \sqrt{\frac{1}{1+\omega^2 R^2 C^2}}$$

and $$V_{co} = V_0 \sqrt{\frac{1}{\omega^2 R^2 C^2 + 1}} \cos(\omega t + \phi)$$

$= V_0$ for $\omega = 0$ since $\phi(\omega=0) = 0$

<u>Impedance</u>: In general we define at frequency ω,

$$\boxed{\text{impedance} \quad Z = \frac{\hat{V}}{\hat{I}}}$$

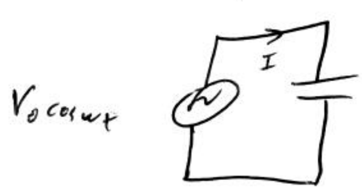
complex amplitudes of V, I :

$V = Re\left(\hat{V} e^{i\omega t}\right)$

$I = Re\left(\hat{I} e^{i\omega t}\right)$

Capacitor:	$Z = \dfrac{1}{i\omega C}$
Inductor:	$Z = i\omega L$
Resistor:	$Z = R$

For a pure capacitor, verify that $Z = \dfrac{1}{i\omega C}$ works:



$\hat{V}_0 = \hat{V}_c$, obviously

$= \hat{Z}_c \hat{I}_c$

$= \dfrac{1}{i\omega C} \hat{I}_c$

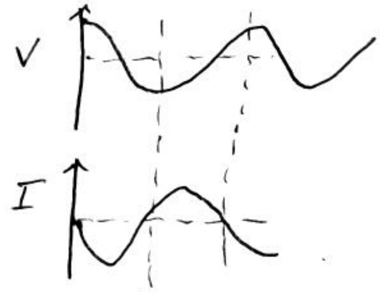
In other words, $V = V_0 \cos \omega t$ (this is given)

and $I = \mathrm{Re}\,(i\omega C\, V_0\, e^{i\omega t})$

$I = -\omega C\, V_0 \sin \omega t$ (since $e^{i\omega t} = \cos\omega t + i \sin \omega t$)

$I = \omega C\, V_0 \cos(\omega t + \frac{\pi}{2})$ $\boxed{\text{same as } C\frac{dV}{dt}}$

So the current leads the voltage by 90°:



(verifies def'n of Z_c)

The ratio of the magnitudes is the reactance,

$$|X_c| = \frac{V_0}{I_0} = \frac{1}{\omega C} \quad \left(\text{actually, } X_c = \frac{-1}{\omega C}\right)$$

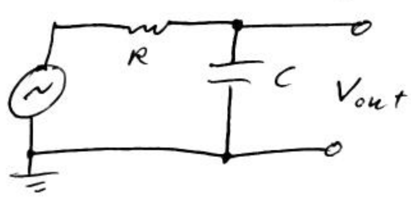
If we instead use an inductor,

$I = \frac{1}{\omega L}\cos(\omega t - \frac{\pi}{2})$, current lags by 90°,

and $X_L = \omega L$

The circuit on pp. RLC-5, RLC-7 can now be recognized fully as a generalized voltage divider:

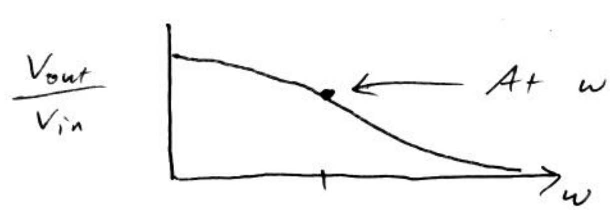
Low-Pass Filter



$$\hat{V}_{out} = \hat{V}_{in}\,\frac{Z_c}{R + Z_c}$$

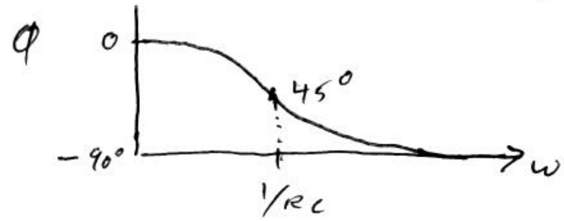
$$V_{out} = V_{in}\,\frac{1}{\sqrt{1 + \omega^2 R^2 C^2}},$$

phase shift $\phi = -\tan^{-1}(\omega RC)$

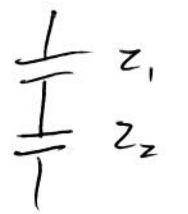


At $\omega = \frac{1}{RC}$, $V_{out} = \frac{V_{in}}{\sqrt{2}} = 0.707\, V_{in}$

We call this the "3 dB point"

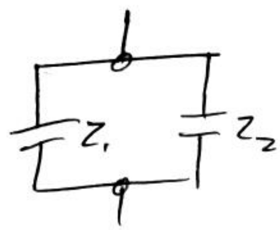


Another Example: capacitors in series & parallel:



$$Z = Z_1 + Z_2 = \frac{1}{i\omega C_1} + \frac{1}{i\omega C_2} = \frac{1}{i\omega}\left(\frac{1}{C_1} + \frac{1}{C_2}\right) = \frac{1}{i\omega C_{eq}}$$

$$\Rightarrow \boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}}$$

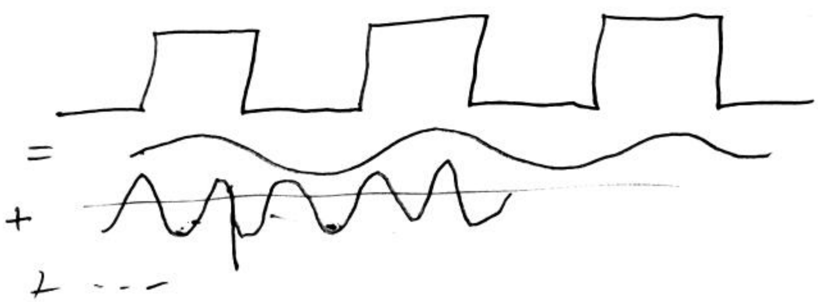


$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

$$i\omega C_{eq} = i\omega (C_1 + C_2), \quad \boxed{C_{eq} = C_1 + C_2}$$

Superposition

But remember, this is for sine wave excitation.
For a general wave would use Fourier's theorem --



Then superpose to get general solution.

For $V_0 e^{i\omega t}$, $\quad \hat{V}_1 = \frac{Z_c}{R + Z_c} \hat{V}_0 = \frac{\frac{1}{i\omega C}}{R + \frac{1}{i\omega C}} \hat{V}_0$

for V_{dc}, $\quad \hat{V}_2 = V_{dc}$

So $\quad \hat{V} = V_{dc} + \frac{1}{1 + i\omega RC} \hat{V}_0$

$$V = \mathrm{Re}\left(V_{dc} + \frac{1}{1 + i\omega RC} \hat{V}_0 e^{i\omega t}\right)$$

$$= V_{dc} + \text{sol'n at } \omega.$$

<u>Power</u> in sinusoidally driven ac circuits:

In real notation,

$$P = IV = \mathrm{Re}(\hat{I})\,\mathrm{Re}(\hat{V})$$

For a device with impedance Z,

$$\hat{V} = V_0 e^{i\omega t}, \quad \hat{I} = \frac{V_0}{Z} e^{i\omega t} = \frac{V_0}{|Z|e^{i\varphi}} e^{i\omega t}$$

so $$P = \frac{V_0^2}{|Z|} \cos\omega t \cos(\omega t - \varphi)$$

The average power over a cycle is then

$$\langle P \rangle = \frac{V_0^2}{|Z|} \frac{1}{2\pi/\omega} \int_0^{2\pi/\omega} \cos\omega t \cos(\omega t - \varphi)\, dt$$

$$\neq \mathrm{Re}(\langle \hat{I}\hat{V} \rangle)$$

$$\langle P \rangle = \frac{V_0^2}{|Z|} \frac{\omega}{2\pi} \int \Big(\underbrace{\tfrac{1}{2}\cos(2\omega t - \varphi)}_{=0 \text{ over cycle}} + \tfrac{1}{2}\cos\varphi \Big)\, dt$$

$$\boxed{P_{avg} = \frac{1}{2}\frac{V_0^2}{|Z|}\cos\varphi = \frac{V_{rms}^2}{|Z|}\cos\varphi}$$

<u>power</u> <u>factor</u> = 0 for pure L, C

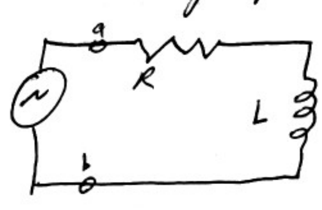
In general, <u>only</u> <u>real</u> <u>Z's</u> <u>consume</u> <u>power</u>.

This can alternatively be written as

$$\boxed{P_{avg} = \frac{1}{2}\mathrm{Re}(V_0^* I_0)} = \frac{1}{2}\mathrm{Re}\left(\frac{V_0^* V_0}{Z}\right) = \frac{|V_0|^2}{2}\mathrm{Re}\frac{1}{Z}$$

$$= \frac{|V_0|^2}{2|Z|}\cos\varphi \text{ as above}$$

<u>Example</u>: LR high-pass filter



Input impedance seen by source is

$$Z = R + i\omega L$$

(Re, Im parts equal at $\omega_0 = \frac{R}{L}$)

$$|Z| = \sqrt{R^2 + \omega^2 L^2}, \quad \varphi = \tan^{-1}\frac{\mathrm{Im}(Z)}{\mathrm{Re}(Z)} = \tan^{-1}\frac{\omega L}{R}$$

Note that $\cos(\tan^{-1} a) = 1/\sqrt{1+a^2}$

so $$\cos\varphi = 1\Big/\sqrt{1 + \frac{\omega^2 L^2}{R^2}} = \frac{R}{\sqrt{R^2 + \omega^2 L^2}}$$

Thus $$P_{avg} = \frac{V_0^2 R}{2}\frac{1}{R^2 + \omega^2 L^2} \text{ approaches } \frac{V_0^2}{2R} \text{ as } \omega \to 0$$

Incidentally, it's easy to show that in general,

$$\mathrm{Re}\, z_1 \, \mathrm{Re}\, z_2 = \frac{1}{2}\left(\mathrm{Re}\,(z_1 z_2) + \mathrm{Re}\,(z_1^* z_2)\right).$$

For a sinusoid, cycle averge = 0

Often power is measured in <u>Decibels</u>:

In comparing two powers P_1 and P_2,

$$dB = 10 \log_{10} \frac{P_2}{P_1} \quad \text{or} \quad 20 \log_{10} \frac{V_2}{V_1}$$

amplitudes

3 dB = factor of 1.995... ≃ factor of 2 in power

6 dB ≃ factor of 2 in voltage

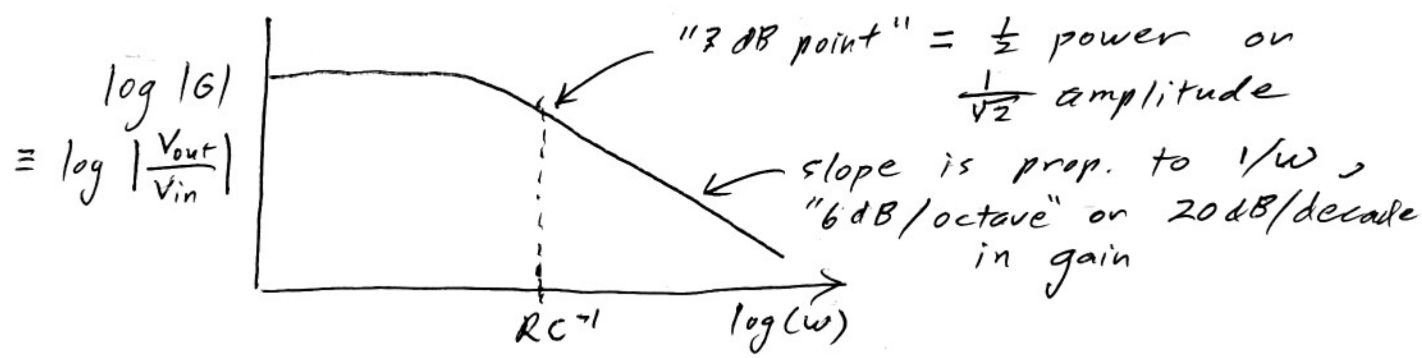
Normally used as a relative measure, but can also be referred to a standard reference point -

dBm are dB relative to 1 mW into 50 ohms

dBV are dB relative to 1 Volt rms

Decibels are very widely used in describing filters and response curves

For an RC low-pass,



Note that in a plot of power, slope would be larger on graph, but would still be described as 6 dB/octave --- dB <u>always</u> refers to <u>power</u>.

These "Bode plots" are used extensively to study response and ϕ shifts.

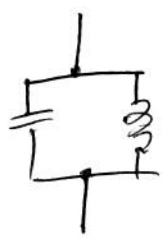
The 6dB/octave is characteristic of any single-pole network, (whose ~~V/I~~ $\hat{V}_{out}/\hat{V}_{in}$ transfer function has a single zero in the denominator). It also inevitably has a phase shift of the type we've described! (complex analyticity).

High-pass filter:

$$\frac{V_{out}}{V_{in}} = \frac{R}{R + \frac{1}{i\omega C}}$$

$$I_{out} = \frac{V_{out}}{R} = V_{in}\,\frac{i\omega C}{1 + i\omega RC}$$

LC & LRC circuits; resonance



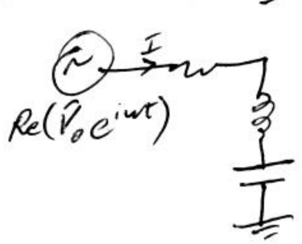
$$Z_{eff} = \frac{1}{\frac{1}{Z_C} + \frac{1}{Z_L}} = \frac{1}{i\omega C + \frac{1}{i\omega L}}$$

$$= \frac{i\omega L}{1 - \omega^2 LC}$$

So V, I are out of phase by $\pm 90°$ and $|Z| = \frac{\omega L}{1-\omega^2 LC} \to \infty$ as $\omega \to \frac{1}{\sqrt{LC}} = \omega_0$

Series LC similar, but $|Z| \to 0$ as $\omega \to \frac{1}{\sqrt{LC}} = \omega_0$.

More realistic is LRC circuit without the discontinuity. As example, look at series case. Monitor I or $V_R = IR$



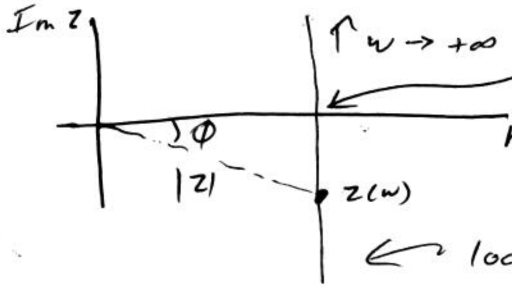
$$Z_{eff} = R + i\omega L + \frac{1}{i\omega C}$$

$$\hat{I} = \frac{\hat{V}_0}{R + i\omega L + \frac{1}{i\omega C}}$$

Writing $Z = |Z| e^{i\phi}$, $|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$

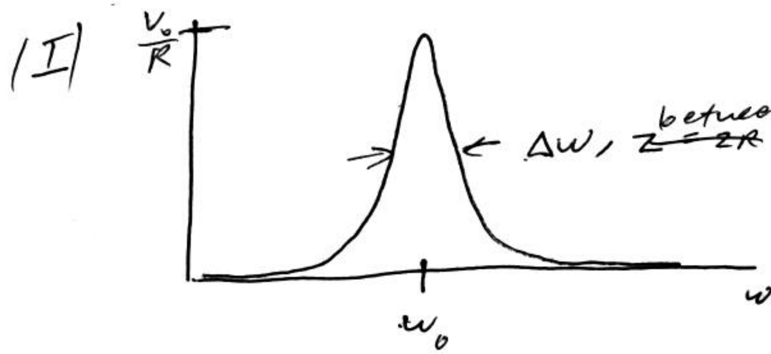
$$\phi = \tan^{-1} \frac{\omega L - \frac{1}{\omega C}}{R}$$

and $I = \frac{V_0}{|Z|} \cos(\omega t - \phi)$



resonance, $\omega_0^2 = \frac{1}{LC}$, $|Z| = R$, $\phi = 0$

$\leftarrow$ locus of possible values of Z = vertical line



$\leftarrow \Delta\omega$, ~~$Z = 2R$~~ between 3dB points where when value = $\frac{\text{peak}}{\sqrt{2}}$

"Quality" factor $Q = \frac{\omega_0}{\Delta\omega} = \frac{f_0}{\Delta f}$, measured between 3-dB points where ~~P~~ $|I| = \frac{1}{\sqrt{2}}$ of peak

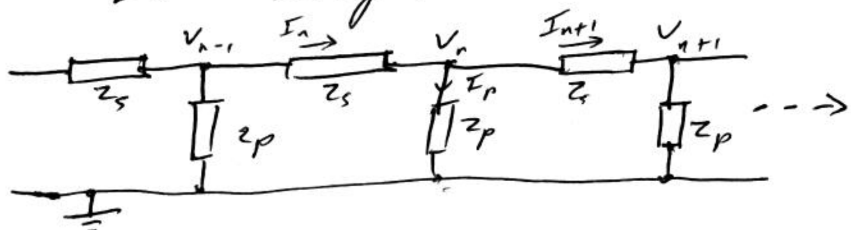
$= \omega_0 L / R$

LC parallel RLC is similar (it's in lab!)

Here we get $\omega_0^2 = \frac{1}{LC}$ again, but

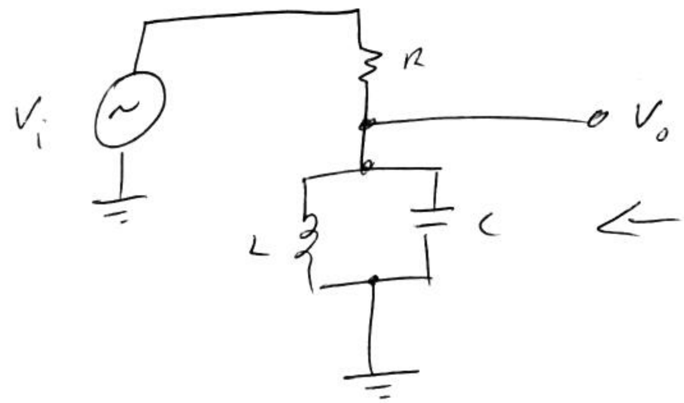
$Q = \frac{R}{L\omega_0}$ (see next page for details)

<u>Transmission</u> <u>Line</u>: To expand horizons, consider an LC string:



Assume Z_s, Z_p are L's & C's. Then $\text{Re}(Z) = 0$ and there's no power dissipation

<u>Detail</u>: Q of RLC in lab 3:



$\leftarrow \; Z_{LC} = \frac{i\omega L}{1-\omega^2/\omega_0^2})$

$\omega_0^2 = \frac{1}{LC}$

$$G = \frac{V_o}{V_i} = \frac{Z}{R+Z} = \frac{i\omega L}{i\omega L + R\left(1-\frac{\omega^2}{\omega_0^2}\right)}$$

$$|G|^2 = \frac{\omega^2 L^2}{\omega^2 L^2 + \left(R\left(1-\frac{\omega^2}{\omega_0^2}\right)\right)^2}$$

$= 1$ at $\omega = \omega_0$

$= \frac{1}{2}$ when $\omega L = R\left(1-\frac{\omega^2}{\omega_0^2}\right)$

$\frac{1}{2}$-power points at $\frac{\omega L}{R}\omega_0^2 = \omega_0^2 - \omega^2$

For $\omega \simeq \omega_0$, $\omega_0^2 - \omega^2 \simeq 2\omega_0(\omega-\omega_0)$, so

$\omega - \omega_0 \simeq \frac{\omega L}{2R}\omega_0 \simeq \frac{L}{2R}\omega_0^2$

Thus $Q = \frac{2\omega_0}{2(\omega-\omega_0)} = \frac{R}{L\omega_0}$ $\left(= \frac{1}{Q} \text{ for series RLC.}\right)$

$\simeq 14$ for $\omega_0 = 2\pi\,(20\text{ KHz})$,

$R = 39K$

$L = 22\,mH$

We also expect no one stage to be "special";
Thus a good guess is,

$$V_n = V_{n-1}\, e^{i\phi} \quad , \text{ constant } \phi \text{ shift / stage.}$$

(like a standing wave, or pure travelling wave)

Circuit eqn's :

① $V_{n-1} - V_n = I_n Z_s$

② $I_n - I_{n+1} = I_p = \dfrac{V_n}{Z_p}$

② × Z_s gives $I_n Z_s - I_{n+1} Z_s = V_n \dfrac{Z_s}{Z_p}$

from ①, $(V_{n-1} - V_n) - (V_n - V_{n+1}) = V_n \dfrac{Z_s}{Z_p}$

$$V_{n-1} + V_{n+1} = \left(2 + \frac{Z_s}{Z_p}\right) V_n$$

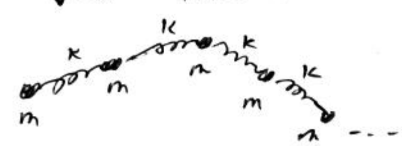
Substituting $V_n = e^{i\phi} V_{n-1}$,

$$e^{i\phi} + e^{-i\phi} = 2 + \frac{Z_s}{Z_p}$$

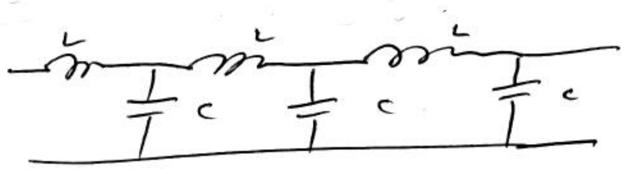
$$2\cos\phi = \text{"}$$

$$\boxed{4\sin^2\frac{\phi}{2} = -\frac{Z_s}{Z_p}}$$

just like :

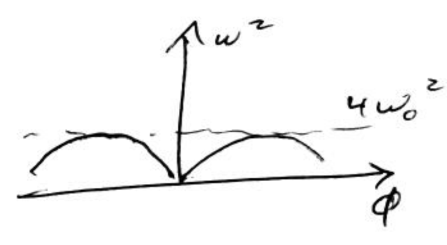


For LC ,



$$\frac{Z_s}{Z_p} = \frac{i\omega L}{1/i\omega C} = -\omega^2 LC$$

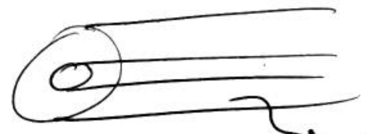
$$\boxed{4\sin^2\frac{\phi}{2} = \frac{\omega^2}{\omega_0^2}} \; , \quad \omega_0^2 = \frac{1}{LC}$$



Transmits only $\boxed{\omega < 2\omega_0}$

When $\omega > 2\omega_0$, Φ is imaginary $\Rightarrow$ exp. attenuation!

Continuous limit:



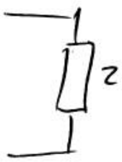
→ cap. C/ℓ per length
ind. L/ℓ " "

Ratio L/C conserved.

To make it look infinite, terminate with Z_{char}:

$$Z = i\omega L + \frac{1}{i\omega C + \frac{1}{Z}}$$

$Z = ?$ Well,



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$$\Rightarrow Z = \frac{i\omega L}{2} + \sqrt{\frac{L}{C} - \left(\frac{\omega L}{2}\right)^2}$$

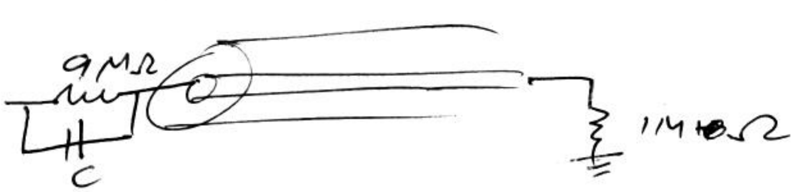
solve for Z

↑ inductor ↑ resistor (value depends on ω!)

Cont. limit: $Z = \sqrt{L/C}$ is a <u>resistor</u>:

looks infinite!

Trickier -- scope probes:



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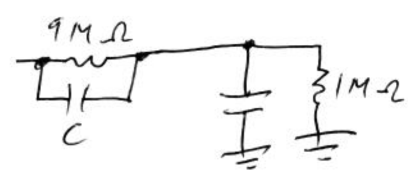

C_{cable} is compensated by a cap. at the scope end of the probe ass'y -- see text. Forms a 10:1 divider at all frequencies!

In a 500 MHz probe, 10-12 components. Deals with L_{cable}, tuning for finite length, etc.