Review (or introduction) to Complex numbers and variables.

A complex number is just an ordered pair (a, b) of real numbers, with special rules for addition, subtraction, etc. To motivate these rules and as an aid to remembering them, we introduce a quantity i, where

$$i^2 = -1 \text{, and write} \qquad (1)$$

$$z = a + ib \qquad (2)$$

z = complex #
a = real part
b = imaginary part

so i is itself a complex #, with $Re(i) = 0$, $Im(i) = 1$

Can also have complex variables or complex functions, by which we just mean they can take on complex values.

Rules for arithmetic

Using (1) and (2), it is easy to see that if $z_1 = a + ib$, $z_2 = c + id$.

Multiplication: $$z_1 z_2 = (a+ib)(c+id) = ac - bd + i(bc + ad)$$

Division: $$\frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{a+ib}{c+id}\left(\frac{c-id}{c-id}\right) = \frac{ac+bd}{c^2+d^2} + i\,\frac{bc-ad}{c^2+d^2}$$

Magnitude :

$$|z|^2 = z z^* = (a+ib)(a-ib)$$
$$= a^2+b^2$$

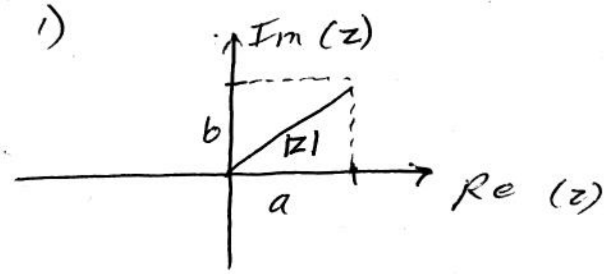
where $z^* \equiv$ complex conjugate

formula: $i \to -i$ everywhere.

So the expression for division above is just,

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\,\frac{z_2^*}{z_2^*} = \frac{z_1 z_2^*}{|z_2|^2}$$

Polar form (round 1)



Can represent a complex number using exponentials to fit this picture directly. First must define the exponential. Do this using a Taylor series,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

So since we know how to do multiplication, can use rhs to define exponentiation:

$$e^{ib} = 1 + ib - \frac{b^2}{2!} - \frac{ib^3}{3!} + \frac{b^4}{4!} + \cdots$$
$$= \left(1 - \frac{b^2}{2!} + \frac{b^4}{4!} + \cdots\right) + i\left(b - \frac{b^3}{3!} + \frac{b^5}{5!} + \cdots\right)$$
$$= \cos b + i \sin b$$

And thus, $e^z = e^{a+ib} = e^a e^{ib} = \cancel{a}\, e^a(\cos b + i \sin b)$

Especially useful is the "phasor", $e^{i\omega t} = \cos\omega t + i \sin\omega t$

rotates ccw in a circle, proj. on Re axis = cosine

Back to starting pt. at $0, 2\pi, 4\pi, \ldots$

Polar form (round 2)

To exploit this relation to circular or oscillatory motion it is often convenient to use the form,

$$z = re^{i\theta}$$

$$\text{then } |z| = \sqrt{re^{i\theta}(re^{-i\theta})} = r \quad \text{(radius)}$$

θ = phase, gives relative size of Re & Im parts

If $z = a + ib = re^{i\theta}$,

$$\boxed{r^2 = a^2 + b^2} \quad \text{and}$$

$$\frac{z}{r} = \frac{a+bi}{\sqrt{a^2+b^2}} = e^{i\theta} = \cos\theta + i\sin\theta$$

$Re_{left} = Re_{rt}$; $Im_{\ell} = Im_r$; so, $\boxed{a = r\cos\theta}$, $\boxed{b = r\sin\theta}$

and $\tan\theta = \frac{b}{a}$; $\boxed{\theta = \tan^{-1}\frac{b}{a}}$

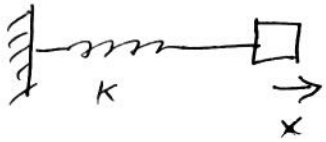
Application of complex #'s.

The idea is to represent the observable quantity of interest as the Re part of a complex variable,

$A = Re\,\hat{A}$; for now, use this notation to distinguish Re part from complex var -- will not use for constants.

Since the complex variable can simultaneously represent two portions of a problem; say, amplitude & phase, the algebra is greatly simplified. Trig functions become exponentials, and are easy to handle. Sometimes, both $Re(\hat{A})$ and $Im(\hat{A})$ are physically significant, but this need not be the case.

<u>Trivial example</u>: undamped harmonic oscillator



$$F = m\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} + w_0^2 x = 0 \quad , \quad w_0 \equiv \sqrt{k/m}$$

We will look for sinusoidal solutions (the only kind, actually).
Rather than substituting a guess in the form,

$$x = A\cos(w't + \phi),$$

then finding A, w', ϕ as usual,
Make the <u>same</u> guess, but in the form,

$$\hat{x} = \hat{c}\, e^{i\hat{w}t} \quad , \quad \hat{c}, \hat{w} \text{ complex}$$

$\hat{c} = Ae^{i\phi}$ gives A and ϕ

$\hat{w} = w' + \cancel{i\beta}$ gives the frequency

zero here, gives damping if present

where the <u>physical</u> solution is taken to be,

$$x = Re(\hat{x})$$

It's certainly OK to write,

$$\frac{d^2 Re(\hat{x})}{dt^2} + w_0^2 Re(\hat{x}) = 0$$

To go further, note that $\frac{d}{dt} e^{i\hat{w}t} = iw\, e^{i\hat{w}t}$ (easy!)

and that $\frac{d}{dt} Re(e^{i\hat{w}t}) = Re\left(\frac{d}{dt} e^{i\hat{w}t}\right)$

(can show by inspecting Taylor series, term by term.

Thus we can solve the problem in complex form, taking Re part only when done:

$$Re\left(\frac{d^2\hat{x}}{dt^2} + w_0^2\hat{x}\right) = 0$$

$$Re\left(-\hat{w}^2\hat{C}e^{i\hat{w}t} + w_0^2\hat{C}e^{i\hat{w}t}\right) = 0$$

$$Re\left((-\hat{w}^2 + w_0^2)\hat{C}e^{i\hat{w}t}\right) = 0$$

This must be true for <u>any</u> choice of time t. So if $-\hat{w}^2 + w_0^2$ is not zero, there will be some t for which $\hat{C}e^{i\hat{w}t}$ is real, obviously violating the condition, and it must be true that

$$\boxed{-\hat{w}^2 + w_0^2 = 0}$$

$$\Rightarrow \hat{w} = w_0, \text{ as expected.}$$

Note that $\hat{C}$ is arbitrary, as the amplitude and phase depend on the initial conditions.

It's easy to incorporate damping and driving terms, also --- the complex notation changes trigonometry into algebra, making things much easier. We will look at this situation a little later.

Anyhow, we have confirmed that solutions exist with the expected form,

$$x = Re\left(Ae^{i\phi}e^{iwt}\right) = Re\left(Ae^{i(wt+\phi)}\right)$$

$$\boxed{x = A\cos(wt + \phi)}$$