Least-Squares Fit to a damped sinusoid

Uses the generalized least-squares fit in MathCad to find the optimal parameters for an exponentially damped sinusoid, allowing for arbitrary amplitude and phase. The offset is assumed to be zero, so if the equilibrium value is non-zero, it should be subtracted from the data..

Last modified by Edward Eyler, Oct. 30, 2005. Tested with Mathcad 11, saved in format of MathCad 8.

First, read a data file containing the measured data, presumably obtained from an A/D converter on a computer. (Note: for this example spreadsheet, evaluation of the first four formulas has been disabled, so that a data file is not needed to display representative results.)

$\text{Data} :=$ A:\Data.tx — Use the data file input component to read the data.

$\text{Data}_0 := \text{Data}_1$ — Throw away the first data point (often not valid).

$t := 0 .. \text{last}(\text{Data})$ — Define an index variable for the time, for later use.

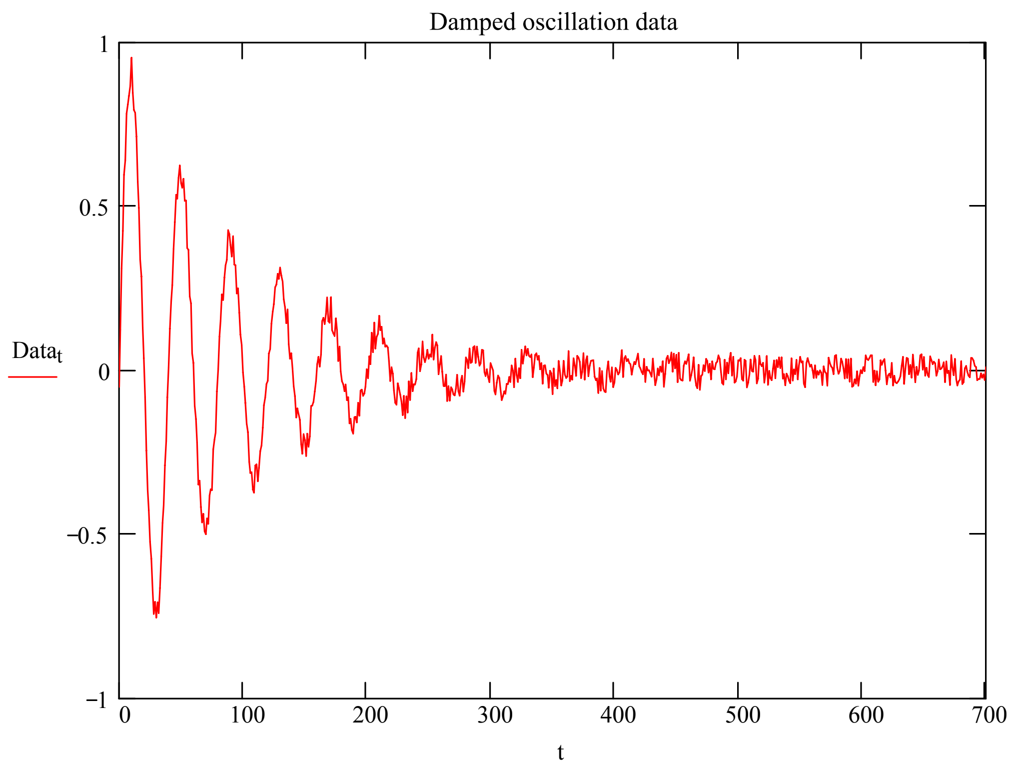
$\text{Data} := \text{Data} - \text{Data}_{\text{last}(\text{Data})}$ — Subtract the offset of the equilibrium position, here assumed to be represented accurately by the last data point.

For evaluation purposes, substitute a calculated damped sine wave, with some random noise added for good measure.

$t := 0 .. 700$ (Length would normally be supplied by the formula above.)

$$\text{Data}_t := \sin\left(2\cdot\pi\cdot\frac{t}{40}\right)\cdot e^{\frac{-t}{100}} + \text{rnd}(0.1) - 0.05$$

Plot the data array to make sure things look reasonable. The plot can also be used to directly estimate the period of oscillation and the exponential decay time constant (the former, by measuring the zero-crossings; the latter, by reading off the maxima for each oscillation, then fitting to an exponential). Note that the "zoom" and "trace" features of MathCad, available by right-clicking on the graph, are extremely useful for direct quantitative measurements of graphical data.



Now fit a damped exponential to the data. We will need an array containing the values of the independent variable, t:

$$tv_t := t$$

Write the fitting function,

$$a \cdot e^{-\gamma \cdot t} \cdot \sin(2 \cdot \pi f \cdot t + b)$$

in terms of a vector of unknown parameters:

$$P_0 \cdot e^{-P_1 \cdot t} \cdot \sin(2 \cdot \pi \cdot P_2 \cdot t + P_3)$$

Define fitting parameters:

$$F(t,P) := \begin{pmatrix} P_0 \cdot e^{-P_1 \cdot t} \cdot \sin(2 \cdot \pi \cdot P_2 \cdot t + P_3) \\ e^{-P_1 \cdot t} \cdot \sin(2 \cdot \pi \cdot P_2 \cdot t + P_3) \\ -P_0 \cdot t \cdot e^{-P_1 \cdot t} \cdot \sin(2 \cdot \pi \cdot P_2 \cdot t + P_3) \\ P_0 \cdot 2 \cdot \pi \cdot t \cdot e^{-P_1 \cdot t} \cdot \cos(2 \cdot \pi \cdot P_2 \cdot t + P_3) \\ P_0 \cdot e^{-P_1 \cdot t} \cdot \cos(2 \cdot \pi \cdot P_2 \cdot t + P_3) \end{pmatrix}$$

This is a vector-valued function whose first component is the function to be fitted and whose remaining components are the partial derivatives with respect to each parameter of that function.

$$Pg := \begin{pmatrix} 1 \\ \frac{1}{111} \\ \frac{1}{45} \\ 0 \end{pmatrix}$$

Enter a vector of guess values for the unknown parameters. They must be reasonably accurate, or the fit will not converge!

Now find the best-fit parameters using a generalized least-squares algorithm:

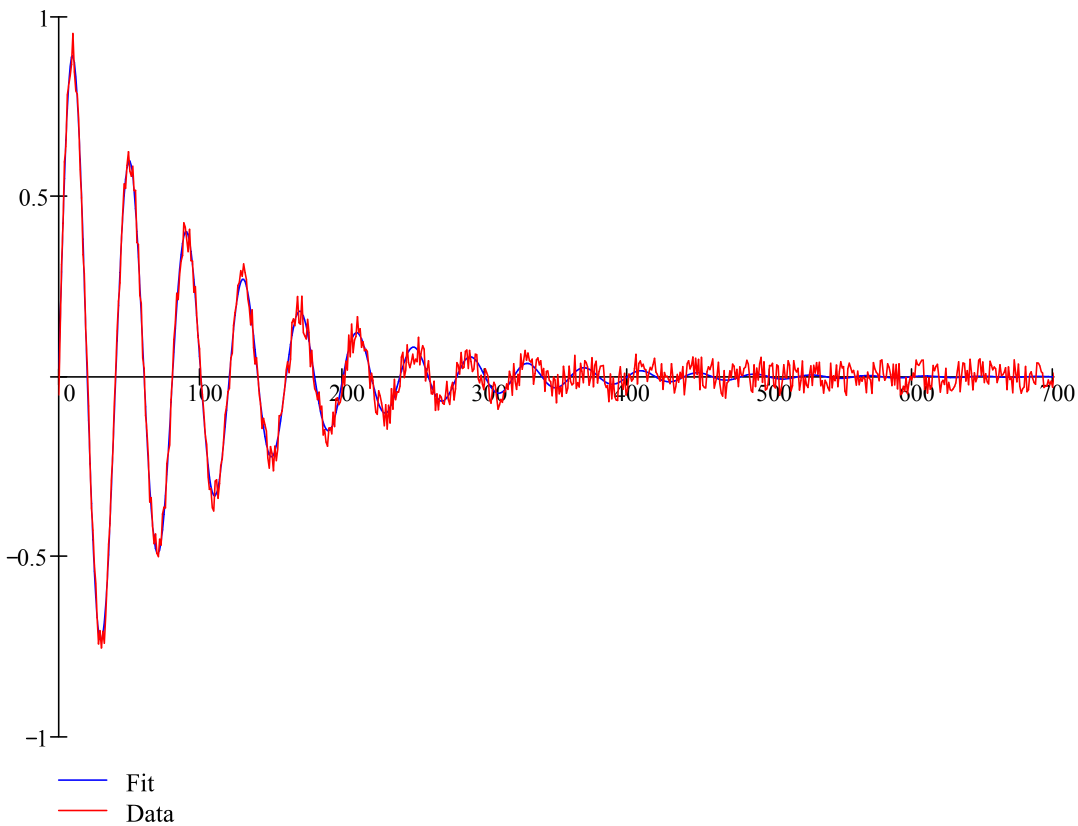
$$Parms := genfit(tv, Data, Pg, F)$$

Here are the results: (warning: for a large array of 5000+ points, convergence is quite slow---it may even require 1-2 minutes.)

$$Parms = \begin{pmatrix} 9.8454 \times 10^{-1} \\ 9.9278 \times 10^{-3} \\ 2.5015 \times 10^{-2} \\ -7.9572 \times 10^{-3} \end{pmatrix}$$

Define a functional form for the fit, then plot the fit vs. the data. In the case of a very good fit, it may be necessary to offset the graph of the fit slightly in order to distinguish the fit from the data.

$f(t) := F(t, Parms)_0$



Finally, plot the residuals, after determining an appropriate vertical scaling range. Note that in most cases, the purely statistical uncertainties in the fit parameters are irrelevant, because obvious systematic deviations will be present (i.e., the plot below will reveal that the residuals are not completely random.)

$$\text{scale} := \max\left(\overrightarrow{|f(tv) - \text{Data}|}\right)\cdot 1.1$$