Preliminary Exam: Statistical Mechanics, Tuesday August 25, 2020. 9:00-12:00

Answer a total of any **THREE** out of the four questions. If a student submits solutions to more than three problems, only the first three problems as listed on the exam will be graded. Students should write their solutions on blank 8.5 by 11 paper or in a blue book, putting their name on each page, the number of the problem and the number of the page in their solution on each page (i.e. 2-1 means first page of problem 2). Also each problem solution should be on a separate set of pages (i.e. not putting parts of two different problems on the same page). At the end of the exam students should scan in their solutions in sequence using a cell phone or a scanner and email them in a file or files to the prelim committee chair philip.mannheim@uconn.edu no later than 15 minutes after the end time of the exam. (It might be easier to transfer the files to a laptop first.) Label both the email header and the file or files with your name and the name of the exam. In the email state which problems you have attempted and state how many pages there are for each of the problems. The chair will immediately check if the emailing is readable or if a resend is required.

Some possibly useful information:

$$\ln N! \approx N \ln N - N + \frac{1}{2} \ln[2\pi N] \text{ as } N \to \infty, \qquad \int_0^1 dx \, x^2 \, (1-x^2)^{3/2} = \frac{\pi}{32},$$

$$\int_0^\infty dx \, x \, \exp(-\alpha x^2) = \frac{1}{2\alpha}, \qquad \int_{-\infty}^{+\infty} dx \, \exp(-\alpha x^2 + \beta x) = \sqrt{\frac{\pi}{\alpha}} \exp(\frac{\beta^2}{4\alpha}) \text{ with } \mathrm{Re}(\alpha) > 0.$$

1. **(a)** Write down the canonical partition function for a monatomic classical ideal gas, and hence derive the ideal gas law using its Helmholtz free energy

   (b) If the temperature of this gas is uniform, derive the condition for hydrostatic equilibrium in a uniform, constant gravitational field and show that the pressure varies with height z according to the barometric formula

   $$P(z) = P(0) \exp(-mgz/kT).$$

   Identify the undefined symbols used in the above equation.

   (c) In an adiabatic ideal gas atmosphere with local density ρ, derive the corresponding formula; i.e., the one in which P/ρ^γ stays constant. Also obtain the variation of the temperature as a function of z.

2. A uniform magnetic field B is applied to a gas of uncharged, spin $1/2$ non-relativistic free particles in a constant volume V, each with magnetic moment μ, obeying Fermi-Dirac statistics. Lining up of the spins produces a magnetic moment density M in this system.

   (a) Set up general expressions for M at temperature $T = 0$ and magnetic field $B > 0$. Determine the magnetic susceptibility of the gas in the limit of zero magnetic field.

   (b) Obtain a closed expression for M at low temperature $T > 0$, in terms of the chemical potential ξ and B.

   Hint: The following low temperature expansion formula may be useful.

   $$\int_0^\infty \frac{\sqrt{\epsilon} \, d\epsilon}{\exp[(\epsilon - \eta)/k_B T] + 1} = \frac{2}{3} \eta^{3/2} [1 + \frac{\pi^2}{8} \left(\frac{k_B T}{\eta}\right)^2 + \ldots]$$

3. A gas of identical non-interacting particles occupies a volume V at a temperature T. The set of allowed particle energies are $\{\epsilon_1, \epsilon_2, ..., \epsilon_j, ..\}$. The total energy E_k of the entire ensemble of particles depends on the number n_j of particles occupying the j-th energy level according to $E_k = \sum_j n_j^{(k)} \epsilon_j$, where the index k indicates a specific set of occupation numbers $k = \{n_1^{(k)}, n_2^{(k)}, ...n_j^{(k)}\}$. Different sets of configurations $\{n_j^{(k)}\}$ can yield the same total energy E_k, but the total number of particles N is fixed: $N = \sum_j n_j^{(k)} = \text{constant}$.

(a) Using the Gibbs partition function with a constant value for the chemical potential μ, viz.

$$Z = \sum_k \exp[-E_k/k_B T] = \sum_k \exp[-\sum_j n_j^{(k)}(\epsilon_j - \mu)/k_B T],$$

which is valid both for Bose and Fermi particles, show that the dispersion $\sigma_j^2 = \overline{(n_j - \bar{n}_j)^2}$ of the mean number of particles $\bar{n}_j$ can be calculated from the formula

$$\sigma_j^2 = -k_B T \frac{\partial \bar{n}_j}{\partial \epsilon_j}.$$

Here k_B is the Boltzmann constant and the bar-symbol means a statistical averaging. Check the result using the standard formulas for the mean occupation numbers for the Fermi-Dirac and Bose-Einstein distributions

(b) Derive the formula for relative fluctuations $\zeta_j = \sigma_j/\bar{n}_j$. For the photon Bose gas with chemical potential $\mu = 0$, express ζ_j as a function of $\bar{n}_j$. Explain, why the relative fluctuations ζ_j do not vanish at large occupation numbers $\bar{n}_j \gg 1$.

Hint: Photon occupation numbers can be very large for photon energies satisfying the condition $\epsilon_j = \hbar\omega_j \ll k_B T$ when $\bar{n}_j \sim k_B T/\hbar\omega_j \gg 1$. Thermal fluctuations of the energy can explain fluctuations of the photon occupation numbers.

4. The classical rotational motion of a large polyatomic molecule can be described as a motion of the three dimensional (3D) rigid rotor with principal moments of inertia I_1, I_2, and I_3. The rotational kinetic energy ϵ_r of such molecules depends on the components of the angular momenta L_j where $j = 1, 2, 3$ as calculated in the center of mass frame:

$$\epsilon_r = \epsilon_r(L_1, L_2, L_3) = \sum_{j=1}^{j=3} \frac{L_j^2}{2I_j}.$$

(a) Find the probability distribution function $\rho_r(\omega_1, \omega_2, \omega_3)$ for the projections of rotational velocities $\omega_j = L_j/I_j$ in the ideal gas of polyatomic molecules. The gas temperature is T. Calculate the mean value of the magnitude of the angular velocity $\bar{\omega}$, if all principal moments of inertia are equal: $I_1 = I_2 = I_3 = I$.

(b) Determine the translational $\rho_{tr}(\epsilon_{tr})$ and rotational $\rho_r(\epsilon_r)$ energy distribution functions of polyatomic molecules moving along a plane surface (the "2D" translational motion). These molecules can rotate only about the axis perpendicular to the plane of translational motion. The corresponding moment of inertia is I. The energy distribution functions have to be normalized to unity.

(c) For the molecules considered in part (b), calculate the distribution function $\rho(\epsilon)$ of the translational-rotational kinetic energy $\epsilon = \epsilon_{tr} + \epsilon_r$.