Preliminary Exam: Statistical Mechanics, Tuesday August 22, 2017. 9:00-12:00

Answer a total of any **THREE** out of the four questions. Use the blue solution books and put the solution to each problem in a separate blue book and put the number of the problem on the front of each blue book. Be sure to put your name on each blue book that you submit. If you submit solutions to more than three problems, only the first three problems as listed on the exam will be graded.

Some possibly useful information:

$$\ln N! \approx N \ln N - N \quad \text{as} \quad N \to \infty, \qquad \int_0^\infty dx\; x\; \exp(-\alpha x^2) = \frac{1}{2\alpha}$$

$$\int_{-\infty}^{+\infty} dx\; \exp(-\alpha x^2 \; + \; \beta x) = \sqrt{\frac{\pi}{\alpha}} \exp(\frac{\beta^2}{4\alpha}) \text{ with } \mathrm{Re}(\alpha) > 0$$

1. (**a**) A metal has heat capacity $c_V = aT + bT^3$ at constant volume. The term with the linear T-dependence is due to the electrons, the term with cubic T-dependence is due to lattice vibrations, and the formula for c_V is valid at low T including both of the temperatures T_1 and T_2 stated below. At the temperature T_1 let the contributions of electrons and lattice vibrations to the total entropy $S_1 \equiv S(T_1)$ be equal. By what factor $c(\alpha)$ does the entropy change if the metal is cooled or heated at constant volume to the temperature $T_2 = \alpha T_1$ with $\alpha > 0$?

   (**b**) Consider a photon gas inside a fixed volume V with the energy $E = b\,T^4$ where b is a positive constant. By what factor does the entropy increase if the photon gas is heated from 1000 K to 3000 K?

2. Consider a solid of volume V made of N non-interacting atoms of spin J at temperature T in a magnetic field $\boldsymbol{B} = B\,\boldsymbol{e}_z$. The magnetic moment of an atom is $\boldsymbol{\mu} = g\mu_0 \boldsymbol{J}/\hbar$ where μ_0 denotes the Bohr magneton, and g is the Landé factor.

   (**a**) Derive the partition function $Z(T,N) = \left(\dfrac{\sinh(J+\frac{1}{2})y}{\sinh \frac{1}{2}y}\right)^N$ where $y = g\mu_0 B/kT$.

   (**b**) Determine the mean value of the z-component $\overline{\mu}_z$ of the magnetic moment of an atom.

   (**c**) The magnetic susceptibility χ is defined as $\overline{M}_z(B) = \chi\,B$ where $\overline{M}_z$ is the mean magnetic moment of the system per unit volume. Determine χ for (**i**) $g\mu_0 B \gg kT$, and (**ii**) $g\mu_0 B \ll kT$.

3. A classical gas of N particles ($N \gg 1$) each with mass m and constant electric dipole moment of magnitude d occupies a volume V. This gas is placed in a time-independent, uniform electric field $\mathbf{E}$ at constant temperature T. Calculate the partition function Z and the free energy $F = -kT \ln Z$ of this gas for two cases:

(**a**) The particles are not interacting (ideal gas).

(**b**) A weak binary interaction exists between each pair of particles, and for any two particles 1 and 2 is described by an effective repulsive potential:

$$U(r_{12}) = U_0 \exp\left(-\frac{r_{12}}{b}\right),$$

where $\mathbf{r}_{12}$ is the vector between 1 and 2, and where U_0 and b are positive constants. For simplicity, the energy of interaction is assumed to be much smaller than the thermal energy, i.e. $U_0 \ll kT$, and also it is assumed that $Nb^3 \ll V$.

Hint: The partition function Z of a weakly interacting classical gas can be calculated using the binary potential approximation:

$$Z = Z_{id} \times \left(1 - \frac{N(N-1)}{2kT} \int_V d^3 r_{12} U(r_{12})\right) = Z_{id} \times \left(1 - \frac{\langle U(r_{12}) \rangle}{kT}\right),$$

where Z_{id} is the partition function of an ideal gas, and $\langle U(r_{12}) \rangle$ is the average energy of each binary interaction.

4. A two-dimensional ($2D$) ideal gas of N fermions each of spin $1/2$ and mass m occupies a macroscopically large region of area A at a temperature T. This region contains N_0 identical zero-range potential wells and each well possesses just one bound state energy level with energy $\epsilon_b = -\epsilon_0$, where $\epsilon_0 > 0$. According to the Pauli principle no more than two fermions may occupy any such energy level.

(**a**) Derive an exact equation for a determination of the chemical potential μ of this Fermi gas.

Hint: The density of states $\rho(\epsilon)$ in the continuous part of the $2D$ energy spectrum is the constant:

$$\rho(\epsilon) = \frac{mA}{\pi \hbar^2},$$

where ϵ is the kinetic energy of a free fermion.

(**b**) In the limit of free fermions where $N_0/N \to 0$, derive the exact analytical formula for the chemical potential $\mu(N, T, A)$, and obtain an expression for the Fermi energy $\epsilon_F = \mu(N, T = 0, A)$.

(**c**) Calculate the number of free (i.e. not bound) electrons N_{fr}, if $N = 2N_0$ and the gas temperature obeys $kT \ll \epsilon_0$, $kT \ll N_0 \pi \hbar^2 / mA$.