STATISTICAL MECHANICS

Preliminary Examination

Tuesday 08/19/2014

09:00 - 12:00 in P-121

Answer a total of **THREE** questions. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book or on sheets of paper stapled together. Make sure you clearly indicate who you are and what is the problem you are solving on each book/sheet of paper. Double-check that you include everything you want graded, and nothing else.

You are allowed to use a result stated in one part of a problem in the subsequent parts even if you cannot derive it. On the last page you will find some potentially useful formulas.

Problem 1. Consider a system of N non-interacting diatomic molecules at temperature T inside a volume V. Let the Hamilton function of a single molecule be given by $H(\vec{p}_1, \vec{p}_2, \vec{r}_1, \vec{r}_2) = (\vec{p}_1^{\,2} + \vec{p}_2^{\,2})/(2m) + \alpha|\vec{r}_1 - \vec{r}_2|^2/2$.

(a) Derive the expression for the classical canonical partition function, and show that it is of the form $Z(T, V, N) = c_N V^N (k_B T)^{9N/2}$ where c_N is a function of N.

(b) Derive an equation of state of the form $f(p, T, V, N) = 0$.

(c) Find the heat capacity at constant volume $C_V(T, V, N)$.

Problem 2. Find the probability distribution $P(\omega_1, \omega_2, \omega_3)$ for the angular velocities ω_i, $i = 1, 2, 3$, in 3D rotation of polyatomic molecules in a classical ideal gas at the temperature T. The respective principal moments of inertia of this molecule are I_1, I_2, I_3. From the distribution P, determine the mean squares of angular velocity $\langle \boldsymbol{\omega}^2 \rangle$ and angular momentum $\langle \mathbf{L}^2 \rangle$ of a molecule.

Problem 3. Consider an ideal Fermi gas of particles with two interconverting species 1 and 2, such as two hyperfine states in an atom. Suppose it is possible to effect the conversion in such a way that an atom picks up an added energy $\Delta\epsilon$ when it moves from species 1 to species 2, and loses the energy $\Delta\epsilon$ in the reverse transition. This could happen, say, if an off-resonant microwave field is transferring atoms between the species. Write the total density of the gas 2ρ, and denote the densities of the individual species by $\rho_{1,2} = \rho \mp \frac{1}{2}\Delta\rho$.

(a) Given the global temperature and pressure, and the energy difference, the equilibrium condition for the two species is $\mu_1 + \Delta\epsilon = \mu_2$. Why?

(b) Show that for a small energy difference $\Delta\epsilon$ and at zero temperature, the density difference equals

$$\Delta\rho = \frac{3\rho}{2\epsilon_F}\,\Delta\epsilon,$$

where ϵ_F is the Fermi energy of a single species at the density ρ.

Problem 4. Collective elementary excitations called *spin waves* or *magnons* determine the low-T specific heat of a spin system that has undergone the ferromagnetic phase transition. For a small wave vector $\mathbf{k}$ the dispersion relation of magnons is $\omega(k) \propto \mathbf{k}^2$. What is the temperature dependence of the heat capacity of magnons at low temperatures?
Hint: As usual in three dimensions, at least at low energies the density of states as a function of the wave number $k = |\mathbf{k}|$ is proportional to k^2.

$$\ln N! \approx N \ln N - N \quad \text{as} \quad N \to \infty$$

$$\int_{-\infty}^{+\infty} dx \exp(-\alpha x^2 + \beta x) = \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta^2}{4\alpha}\right) \quad \text{with} \quad \mathrm{Re}(\alpha) > 0$$

$$\int_0^{\infty} dx\; x \exp(-\alpha x^2) = \frac{1}{2\alpha}$$

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + i \frac{p}{m\omega}\right)$$