Preliminary Examination: Statistical Mechanics, 08/21/2012

Answer a total **THREE** questions out of **FOUR**. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book, or on consecutively numbered sheets of paper stapled together. Make sure you clearly indicate who you are and the problem you are solving. Double-check that you include everything you want graded, and nothing else.

Possibly Useful Information

$$\ln N! \approx N \ln N - N \quad \text{as} \quad N \to \infty$$

$$\int_{-\infty}^{+\infty} dx \exp(-\alpha x^2 + \beta x) = \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta^2}{4\alpha}\right) \quad \text{with} \quad \mathrm{Re}(\alpha) > 0$$

$$\int_0^\infty dx\; x \exp(-\alpha x^2) = \frac{1}{2\alpha}$$

1. Consider a system composed of a very large number $N \gg 1$ of non-interacting distinguishable particles, each of which can only be in one of two states with energies 0 and ε. Let E be the total energy of the system.

    a) Calculate the number of states $\Omega(E)$ of the system as a function of $E/\varepsilon N$.

    b) Compute and plot the entropy per particle S/Nk_B as a function of $E/\varepsilon N$.

    c) What is the temperature T of this system? Plot the temperature as a function of $E/\varepsilon N$.

    d) What is the probability to have a particle in a state with energy ε? Express the final result in terms of the system temperature T and ε.

2. For a non-relativistic ($\varepsilon(p) = p^2/2m$, where p is the momentum and m is the mass at rest) and a relativistic ($\varepsilon(p) = cp$, where c is the speed of light) three dimensional electron gas at $T = 0\,\mathrm{K}$, compute the following.

    a) the average energy per particle $\langle \varepsilon \rangle$

    b) the electron gas pressure P

    Hint: Use the thermodynamic relation

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P$$

3. Assume that the universe can be approximated by a spherical cavity with impenetrable walls and having radius 10^{26} m. If the temperature inside the cavity is $3\,\mathrm{K}$, estimate the total number of photons in the universe and the total energy of these photons.

    Hint:

$$\int_0^\infty \frac{x^2}{\exp(x)-1}dx \approx 2.4$$

$$\int_0^\infty \frac{x^3}{\exp(x)-1}dx = \frac{\pi^4}{15}$$

4. Use the semi-classical limit of statistical mechanics to study N indistinguishable particles in a three-dimensional isotropic harmonic oscillator trap with the potential $V(r) = \frac{1}{2} m\omega^2 r^2$.

a) Show that the canonical partition function is

$$Z = \frac{1}{N!}\left(\frac{k_B T}{\hbar\omega}\right)^{3N}$$

b) Show that the entropy is

$$S = K + 3Nk_\mathrm{B}\ln\left(\frac{k_\mathrm{B}T}{\hbar\omega}\right)$$

where the constant K depends neither on temperature nor on the oscillator frequency.

c) Show that the maximum of the phase space density ρ [such that the number of atoms in a small joint volume of real space ΔV and momentum space ΔV_p is $\Delta N = \rho\,\Delta V\,\Delta V_p$], obviously at $\mathbf{r} = 0$ and $\mathbf{p} = 0$, equals

$$\rho_M = N\left(\frac{\omega}{2\pi kT}\right)^3 .$$

d) The dimensionless parameter that determines whether a Bose-Einstein condensate appears in a (nearly ideal) gas is the number of atoms in a phase space element of the size $\Delta V\,\Delta V_p = (2\pi\hbar)^3$. Is it possible to begin in a state with no Bose-Einstein condensate present, and make the condensate appear by varying the trapping frequency ω adiabatically?