Preliminary Exam: Statistical Mechanics 8/23/2011, 9:00-12:00

Answer a total of **THREE** questions. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set. Be sure to put your name on each book and on each sheet of paper that you submit. Some possibly useful information:

$$\int_0^\infty dx\; x^n e^{-ax} = \frac{n!}{a^{n+1}}, \qquad \int_0^\infty dx\; e^{-a^2x^2} = \frac{\pi^{1/2}}{2a}, \qquad \int_0^\infty dx\; x e^{-a^2x^2} = \frac{1}{2a^2}, \quad \ln(n!) \to n\ln n - n \text{ as } n \to \infty$$

S.1 Consider a grand canonical ensemble of a relativistic free gas of bosons with the dispersion relation $\epsilon(\mathbf{k}) = \hbar ck$.

(a) Show that below a critical temperature T_c the gas is in the Bose condensed state. Find the functional dependence of T_c on the density of gas.

(b) Find the fraction of particles that are in the ground state as a function of T/T_c.

(c) Phonons are quantum excitations of lattices in solids. To a good approximation they behave like relativistic bosons with a velocity c equal to the velocity of sound in the solid. However, unlike such bosons, one does not have control over the number of phonons in a solid. This number is determined dynamically to minimize the free energy of the solid. As a result, the chemical potential for phonons vanishes, $\mu = 0$. Given this information, find the number of phonons in a solid as a function of temperature T. Thus explain why phonons do not Bose condense. Take the temperatures of interest to be small, so that you can disregard modifications of the linear dispersion relation that are important for high-momentum phonon modes.

S.2 Let us study an ideal gas of N massive particles in a potential well of the form $V(r) = Kr^\alpha$ in D-dimensional space (with $K > 0$, $\alpha > 0$ and $D > 0$) in the classical limit of statistical mechanics.

(a) Show that the canonical partition function is of the form

$$Z = A\,T^{DN/2+DN/\alpha}\,,$$

where A is some constant independent of temperature.

(b) Show that the heat capacity is given by

$$C = \left(\frac{1}{2} + \frac{1}{\alpha}\right) ND k_{\mathrm{B}}\,,$$

where k_{B} is Boltzmann's constant.

Hint: The area of the surface of a sphere with radius r in D-dimensional space is

$$\Omega_{\mathrm{D}}(r) = \frac{2\pi^{D/2}}{\Gamma(D/2)}\, r^{D-1}\,,$$

where $\Gamma(x)$ is the usual gamma function.

S.3 For the extremely relativistic ideal Fermi gas (just like for relativistic bosons) one may write one-particle energy as $\epsilon(\mathbf{k}) = \hbar c k$, where $k = |\mathbf{k}|$ is the absolute value of the wave vector $\mathbf{k}$ related to momentum $\mathbf{p}$ by $\mathbf{p} = \hbar \mathbf{k}$. Find (i) the Fermi energy, (ii) the pressure, and (iii) the average energy per particle; all of this, of course, at zero temperature.

S.4 A rectangular box has a partition that divides the box into two equal regions labeled L (left) and R (right), each of volume $V/2$. The box contains an ideal gas of N identical classical particles, each of mass m. Initially all the particles are in the region L and they are in thermal equilibrium at a temperature T. The partition is then suddenly removed so that the gas can expand to fill the whole box.

(a) What is the resulting temperature of the system?

(b) After the gas was allowed to expand, what is the probability that the L and R regions will each contain exactly $N/2$ particles?

(c) After the gas was allowed to expand, what is the probability that all N particles will nevertheless be found the L region?

(d) What is the change in the entropy of the system caused by the removal of the partition?