Prelim and Course Exam: Statistical Mechanics, Monday December 14, 2020. 8:00am-11:00am

For both the course final and for the prelim answer the same **THREE** out of the four questions. If you submit solutions to all four then the three to be graded will be picked at random. Students should write their solutions on blank 8.5 by 11 paper or in a blue book, putting their name on each page, the number of the problem and the number of the page in their solution on each page (i.e. 2-1 means first page of problem 2). Also each problem solution should be on a separate set of pages (i.e. not putting parts of two different problems on the same page). At the end of the exam students should scan in their solutions in sequence using a cell phone or a scanner and email them in a file or files (ideally pdf) to the prelim committee chair philip.mannheim@uconn.edu no later than 15 minutes after the end time of the exam. (It might be easier to transfer the files to a laptop first.) Label both the email header and the file or files with your name and the name of the exam. In the email state which problems you have attempted and state how many pages there are for each of the problems. The chair will immediately check if the emailing is readable or if a resend is required.

1. Verify the thermodynamical identity

$$\left(\frac{\partial N}{\partial V}\right)_{S,\mu} = \left(\frac{\partial p}{\partial \mu}\right)_{S,V}$$

2. (a) A preliminary exercise in geometry: Show that the volume of the pyramid that the plane $x+y+z=a$ ($a>0$) cuts from the first octant $x\geq 0$, $y\geq 0$, $y\geq 0$ of the cartesian coordinate system equals $V=\frac{1}{6}a^3$.

   (b) Consider an isotropic harmonic oscillator in the limit when the oscillator energy $\hbar\omega$ may be regarded as "small." Show that the density of one-particle energy eigenstates $\mathcal{D}(\epsilon)$, such that there are $dN(\epsilon)=d\epsilon\,\mathcal{D}(\epsilon)$ states in the energy interval $[\epsilon,\epsilon+d\epsilon)$, equals

$$\mathcal{D}(\epsilon)=\frac{\epsilon^2}{2(\hbar\omega)^3}.$$

   (c) Find the Fermi energy for N identical fermions with spin s in such a harmonic oscillator potential.

3. A centrifuge is basically a cylinder with some radius R that is being rotated at the angular velocity ω about its axis. When churned long enough, normal (not superfluid) matter inside settles to thermal equilibrium in a frame rotating with the cylinder. On the other hand, transformation to a rotating frame adds a term $-\boldsymbol{\omega}\cdot\mathbf{L}$ to the Hamiltonian, where $\boldsymbol{\omega}$ is the angular velocity vector and $\mathbf{L}$ is the angular momentum with respect to a point on the axis of rotation. Based on these observations, find the equilibrium density at temperature T for a nearly ideal gas of molecules with mass m inside the centrifuge, given that the density without the rotation would be n. Assume that the centrifuge is sealed, i.e., the number of molecules is fixed.

4. Consider a one-dimensional chain of Ising spins (values ± 1) whose interaction attempts to turn them antiparallel. It is convenient to imagine that there are two interlaced lattices of spins s_i and S_i, $2N$ total; say, $s_1, S_1, s_2, S_2, \ldots, s_N, S_N$. The Hamiltonian is written

$$\mathcal{H} = \tfrac{1}{2}\varepsilon \sum_i (s_i S_i + S_i s_{i+1}) - H \left(\sum_i s_i + \sum_i S_i \right) .$$

The sum in effect runs over the pairs of nearest-neighbor spins s_i and S_i, $2N$ terms, $\varepsilon > 0$ is the coupling coefficient of the spin-spin interaction, and H is a constant magnetic field.

(a) Derive the mean-field Hamiltonian in terms of the averages of the spins $s = \langle s_i \rangle$ and $S = \langle S_i \rangle$,

$$\mathcal{H}_{MF} = (\varepsilon S - H) \sum_i s_i + (\varepsilon s - H) \sum_i S_i - N \varepsilon s S .$$

(b) In the mean-field approximation derive the expression for the Gibbs free energy

$$\frac{G}{N} = -\varepsilon s S - kT \ln\{2 \cosh[\beta(\varepsilon S - H)]\} - kT \ln\{2 \cosh[\beta(\varepsilon s - H)]\} .$$

What you get from statistical mechanics thinking is the Helmholtz free energy is thermodynamically actually the Gibbs free energy, with T and H as the natural variables.

(c) Show (in the mean-field approximation, of course) that, at $H = 0$, when the temperature is lowered below $T = T_c = \varepsilon / k$, an *antiferromagnetic* phase transition could take place, in which the alternating spins attain non-zero expectation values with opposite signs.

(d) While the antiferromagnetic phase found in part (c) is an extremum of G with respect to the variables s and S, the extremum is a saddle point not a minimum. Show this, for simplicity in the limit of very low temperature. Unfortunately for the model, the antiferromagnetic state is not thermodynamically stable.