Preliminary Exam: Statistical Mechanics, Tuesday January 14, 2020. 9:00-12:00

Answer a total of any **THREE** out of the four questions. Put the solution to each problem in a **SEPARATE** blue book and put the number of the problem and your name on the front of each book. If you submit solutions to more than three problems, only the first three problems as listed on the exam will be graded. **Some possibly useful information**:

$$\ln N! \approx N \ln N - N + \frac{1}{2}\ln[2\pi N] \quad \text{as} \quad N \to \infty, \qquad \int_0^1 dx\, x^2\, (1-x^2)^{3/2} = \frac{\pi}{32},$$

$$\int_0^\infty dx\ x\ \exp(-\alpha x^2) = \frac{1}{2\alpha}, \qquad \int_{-\infty}^{+\infty} dx\ \exp(-\alpha x^2\ +\ \beta x) = \sqrt{\frac{\pi}{\alpha}}\exp(\frac{\beta^2}{4\alpha}) \text{ with } \mathrm{Re}(\alpha) > 0.$$

1. When two different species of "molecules" are mixed, a generic *entropy of mixing*

$$\Delta S = -k\left(N_1 \ln\frac{N_1}{N_1+N_2} + N_2 \ln\frac{N_2}{N_1+N_2}\right)$$

arises. From now on, assume that one of the species is much more abundant than the other, $N_1 \gg N_2$. There are varying circumstances such as chemical reactions, but by default in the limit $N_2 \to 0$, the entropy of mixing dominates in the free energy over the effects of the interactions between the molecules of the two species, and for the mixture we have $G(T,p,N_1,N_2) \simeq G_1(T,p,N_1) + G_2(T,p,N_2) - T\,\Delta S(N_1,N_2)$.

   (**a**) Show that for given T, p, the chemical potential of species 1 in the mixture, $\bar{\mu}_1(T,p)$, is related to the chemical potential without the mixing species, $\mu_1(T,p)$, by $\bar{\mu}_1(T,p) \simeq \mu_1(T,p) - kTN_2/N_1$.

   (**b**) Now call species 1 "solvent" and species 2 "solute." Suppose the mixture is separated from a container with pure solvent by a semipermeable membrane which lets solvent molecules through, but not solute molecules, and which can withstand pressure without moving. What are the conditions for thermal equilibrium of the solvent molecules between the two sides?

   (**c**) Show that the pressure of the mixture is higher than the pressure of the pure solvent by the *osmotic pressure* $\Delta p = kTN_2/V$. It is as if the solute were an ideal gas added to the solvent, and exerts the corresponding pressure on the membrane.

2. The local-density approximation of thermodynamics states that the sum of the external potential energy per particle $V(\mathbf{r})$ and the local chemical potential $\mu(\mathbf{r})$ (calculated as if the gas with the given density $n(\mathbf{r})$ were infinite) equals a constant, the global chemical potential μ. Let us study a cloud of zero-temperature, single-component, N-atom Fermi gas trapped in the harmonic oscillator potential $V(\mathbf{r}) = \frac{1}{2}m\omega^2\mathbf{r}^2$ using the local-density approximation. "Single component" means that there is no spin degeneracy; for instance, it could be that only one z-component of the angular momentum is confined by the trapping potential.

   (**a**) Show that the density of the gas is of the form $n(r) = n(0)[1-(r/R)^2]^{3/2}$, where R is the radius of the cloud and $n(0)$ is the central density.

   (**b**) Show that the central density and the radius are related by $\frac{1}{8}\pi^2 R^3 n(0) = N$.

   (**c**) Show that the radius equals $R = (48N)^{1/6}\sqrt{\dfrac{\hbar}{m\omega}}$.

   (**d**) Show that the central density is proportional to $\sqrt{N}$.

3. N non-interacting quasi-particles form an ideal gas inside a crystal material of volume V. The dispersion law $\epsilon(\mathbf{p}) = \mathbf{s} \cdot \mathbf{p} = s_x p_x + s_y p_y + s_z p_z$ describes the relation between the quasi-particle energy ϵ and momentum $\mathbf{p}$. The constant vector $\mathbf{s}(s_x, s_y, s_z)$ is the quasi-particle velocity vector specifically oriented with respect to the crystal axis $i = (x, y, z)$. The gas temperature T is high enough that the quasi-particles obey classical Boltzmann statistics.

   (**a**) Calculate the partition function Z and free energy $F = -k_B T \ln Z$, if the absolute value of the quasi-particle momentum $p = \sqrt{\sum_i p_i^2}$ is restricted by the spherically symmetric condition $p \leq p_D$.

   (**b**) Calculate the free energy F of the above quasi-particle gas, if the values of the momentum projections p_i are confined to the box $|p_i| \leq p_D$.

   (**c**) Determine an average quasi-particle energy $< \epsilon >$ using the free energy F derived in part (**b**), and find an asymptotic expression for $< \epsilon >$ if $s_x \to 0$, $s_y \to 0$ and $s_z \to s$, where s is a positive constant.

4. An ideal gas of Bose particles at temperature T is trapped inside a macroscopically large volume V. The dispersion relation between the particle energy ϵ and momentum $\mathbf{p}$ is $\epsilon(p) = b\, p^\alpha$, where b and α are positive constants.

   (**a**) Calculate the partition function Z and show that relation between the total number of trapped particle N and the chemical potential μ is given by the expression:

$$N = k_B T \frac{\partial \ln Z}{\partial \mu} = \frac{1}{\exp(-\beta\mu) - 1} + \frac{4\pi V}{h^3 \alpha b^{3/\alpha}} \int_0^\infty d\epsilon \frac{\epsilon^{3/\alpha - 1}}{\exp[\beta(\epsilon - \mu)] - 1},$$

   where $\beta = 1/k_b T$. The first term in the sum represents an average number of particles $N_g(\epsilon = 0)$ in the ground state $\epsilon = 0$, and the second term is the number of particles $N_{ex}(\epsilon > 0)$ in excited states $\epsilon > 0$.

   (**b**) Find the critical temperature of Bose condensation T_c in the thermodynamic limit, when the gas density $n = N/V$ is held constant while $N \to \infty$ and $V \to \infty$.

   Hint: The integral required for this solution

$$\frac{1}{\Gamma(y)} \int_0^\infty \frac{x^{y-1}}{\exp(x) - 1} dx = \zeta(y)$$

   can be expressed via the gamma function $\Gamma(y)$ and the Riemann zeta function $\zeta(y) = \sum_{k=1}^\infty k^{-y}$ for all $y > 1$. For $y \leq 1$, the integral diverges.

   (**c**) Determine the α-values for which Bose condensation can occur as a phase transition at a non-zero critical temperature.