STATISTICAL MECHANICS

Preliminary Examination

Tuesday 01/15/2013

9:00am - 12:00pm in P-121

Answer a total of **THREE** questions. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book or on sheets of paper stapled together. Make sure you clearly indicate who you are and what is the problem you are solving on each book/sheet of paper. Double-check that you include everything you want graded, and nothing else.

On the last page you will find some potentially useful formulas.

Problem 1. For an ideal spin-0 Bose gas in *two* spatial dimensions, the chemical potential may be found analytically in closed form in terms of the temperature T and (area) density $n = N/A$. Find it.

Problem 2. Consider N identical noninteracting atoms of mass m, each having angular momentum $J = 1$. The system occupies volume V, is in a thermal equilibrium at temperature T, and is subjected to the magnetic field $\mathbf{H} = (0, 0, H)$. The magnetic dipole moment associated with each atom is $\mu = -g\mu_B\mathbf{J}$, where g is the gyromagnetic ratio and μ_B is the Bohr magneton.

(a) For an atom in this system list all possible values of μ_z, the magnetic moment along the magnetic field $\mathbf{H}$, and the corresponding magnetic energy, $U = -g\mu_B m_z H = -\mu_z H$, for each quantum state.

(b) Calculate the partition function and Helmholtz free energy of the system of atoms in the magnetic field.

(c) Determine the average value of the magnetic moment $\langle\mu_z\rangle$ and the magnetization of the system, M. Sketch the dependence of M on the external magnetic field H.

Problem 3. For an ideal photon gas at temperature T occupying volume V the entropy is

$$S = \frac{1}{T}\sum_i \frac{\hbar\omega_i}{\exp(\hbar\omega_i/\tau) - 1} - k_B\sum_i \ln\left(1 - \exp\left(-\hbar\omega_i/\tau\right)\right)$$

where ω_i is the angular frequency of the i^{th} mode and $\tau = k_B T$ is the effective temperature.

(a) Calculate the Helmholtz free energy, $F = U - TS$, of the photon gas.

(b) Using the expression for the Helmholtz free energy, show that the isothermal work done by the gas is

$$dW = -\hbar\sum_i \langle n_i\rangle \frac{d\omega_i}{dV} dV$$

Hint: use the relation $P = -(\partial F/\partial V)_T$

(c) Show that the radiation pressure is equal to one-third of the average energy density

$$P = \frac{1}{3}\frac{U}{V}$$

(d) (extra credit) Apply the equation for dW to show that for a non-relativistic ideal Fermi gas the pressure is

$$P = \frac{2}{3}\frac{U}{V}$$

Problem 4. A classical particle in one dimension with mass m is connected to two springs, each with spring constant K (see Fig. 1). It is in equilibrium with a heat bath at temperature T.

(a) Show that the classical Hamiltonian of this particle is

$$H(p, x) = \frac{p^2}{2m} + Kx^2$$

where x is the displacement from the equilibrium position.

(b) Calculate the partition function and Helmholtz free energy of this particle in low ($KL^2/k_BT \gg 1$) and high ($KL^2/k_BT \ll 1$) temperature limits. Hint:

$$\text{erf}(a) = \frac{2}{\sqrt{\pi}} \int_0^a \exp(-t^2/2)dt, \;\; \text{erf}(a) \approx 1 (a \gg 1) \text{ and } \text{erf}(a) \approx 2a/\sqrt{\pi} (a \ll 1)$$

(c) What is the mean-square displacement from the equilibrium position due to thermal fluctuations, $\langle x^2 \rangle$, in the low and high temperature limits?

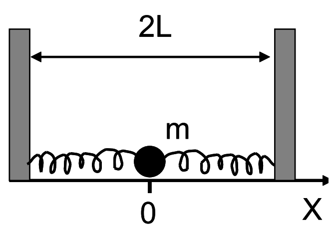

Figure 1: Particle in 1d attached to springs with spring constant K.

$$\ln N! \approx N \ln N - N \quad \text{as} \quad N \to \infty$$

$$\int_{-\infty}^{+\infty} dx \exp(-\alpha x^2 + \beta x) = \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta^2}{4\alpha}\right) \quad \text{with} \quad \mathrm{Re}(\alpha) > 0$$

$$\int_0^{\infty} dx\ x \exp(-\alpha x^2) = \frac{1}{2\alpha}$$

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + i \frac{p}{m\omega}\right)$$