Preliminary Examination: Statistical Mechanics, 1/10/2012

Answer a total **THREE** questions out of **FOUR**. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book, or on consecutively numbered sheets of paper stapled together. Make sure you clearly indicate who you are and the problem you are solving. Double-check that you include everything you want graded, and nothing else.

Possibly Useful Information

$$\ln N! \approx N \ln N - N \quad \text{as} \quad N \to \infty$$

$$\int_{-\infty}^{+\infty} dx\ \exp(-\alpha x^2 + \beta x) = \sqrt{\frac{\pi}{\alpha}} \exp(\frac{\beta^2}{4\alpha}) \text{ with } \mathrm{Re}(\alpha) > 0, \quad \int_0^{\infty} dx\, x\ \exp(-\alpha x^2) = \frac{1}{2\alpha}$$

SM1. Suppose that instead of photons, blackbody radiation were composed of a single species of neutrinos. The neutrino is a spin-1/2 particle like an electron, with zero electric charge. Without worrying about the details of the reactions that neutrinos undergo, suppose that they can be freely created and destroyed such that they maintain thermal equilibrium with the walls of a cavity. Treat the neutrinos as a grand canonical ensemble of free particles of mass m, with chemical potential $\mu = -mc^2$.

(a) Show that the heat capacity per unit volume reduces to the following form at low temperature, where the neutrinos are non-relativistic and fermion quantum statistics reduce to classical Boltzmann statistics.

$$c_v = \frac{1}{V}\left.\frac{dU}{dT}\right|_V = \frac{4k_B}{\lambda^3}e^{\beta\mu}\left[(\beta\mu)^2 - \frac{3}{2}\beta\mu\right]$$

where

$$\lambda = \sqrt{\frac{h^2\beta}{2\pi m}}$$

(b) What is the heat capacity per unit volume of a cavity at asymptotically high temperatures, where the neutrinos are effectively massless?

The following definite integrals may be of use in solving this problem.

$$\int_0^\infty x^2 \exp\left(-\frac{x^2}{2\sigma^2}\right)dx = \sqrt{\frac{\pi}{2}}\sigma^{\frac{3}{2}} \qquad \int_0^\infty \frac{x^3 dx}{e^{\beta x}+1} = \frac{7\pi^4}{120\beta^4}$$

SM2. (a) By studying the classical limit (e.g., asymptotically high temperature) of the spinless ideal massive Bose gas, show that the chemical potential of a monatomic classical ideal gas is

$$\mu = k_B T \ln n\lambda^3; \quad n = \frac{N}{V},\ \lambda = \sqrt{\frac{2\pi\hbar^2}{mk_BT}}\,.$$

(b) Starting from the result of part (a), find the absolute entropy of the classical ideal gas.

(c) In the high temperature/low density limit it is occasionally possible to calculate the absolute value of entropy explicitly, like in the above example. Moreover, it may be possible to track the changes of the entropy of a substance (from temperature dependence of specific heat and latent heats in phase transitions) from this limit all the way down to effectively zero temperature. One can then infer the absolute entropy at zero temperature. For water it is $\ln(3/2)\, k_B$ per molecule, a resounding contradiction with the naive form of the third law. What does this result say about the ground state of water?

SM3. Consider the mean-field Hamiltonian

$$\mathcal{H} = -2Jr\sum_i s_i^z \langle s \rangle - mB\sum_i s_i^z$$

for a lattice of N spin 1/2 particles in a magnetic field B. Here r represents the number of nearest neighbors and $\langle s \rangle$, the average value of s_i^z with m and J being constants.

(a) Find the single site partition function $\mathcal{Z}_i$.

(b) Obtain a self-consistency condition for the average spin $\langle s \rangle$ at temperature T.

(c) Graphically or otherwise, show that a nonzero solution exists for $\langle s \rangle$ in (b) at $B = 0$ when $T_c \equiv Jr/2k > T$.

(d) Find the zero-field susceptibility χ when $T > T_c$. (χ has the so-called Curie-Weiss behavior.)

SM4. A single adsorption site has two energy levels ($\epsilon_1 < \epsilon_2$) separated by an energy gap Δ. The gas above the site contains identical bosons. Suppose the adsorption site can be <u>unoccupied</u>, or occupied by <u>one</u> or <u>at most 2</u> bosons. When the site is doubly occupied, there is an extra interaction energy u, while the energy of an unoccupied site is zero.

(a) Enumerate the singly and doubly occupied states. Give the energies and degeneracies.

(b) Compute the grand partition function for the single site.

(c) Give an expression for the <u>mean occupancy</u> of the site in equilibrium with the gas at temperature T. A chemical potential appears in your expression. What is its interpretation? What information do you need to infer its value? (No calculations here.)