Prelim Exam: Statistical Mechanics, Monday May 3, 2021. 8:00am-11:00am

Answer **THREE** out of the four questions. If you submit solutions to all four then the three to be graded will be picked at random.

To take the prelims remotely students will need a good internet connection, a computer with a camera, a cell phone with a camera, and sufficient cell phone data capacity to switch to the data line on the cell phone if the wifi fails. Also students should keep their cell phone fully charged in case of power outages. Students should have the webex app on both their computer and cell phone. Each webex link will open at 7:45am. The computer camera will only be needed to check each student's ID prior to the start of the exam. The exam will be emailed to each participant at the starting time of the exam. Students should immediately download the exam to their computer and cell phone (and even print it out if they can) in case they lose the internet connection.

Students should write their solutions on blank 8.5 by 11 paper, putting their name on each page, the number of the problem and the number of the page in their solution (i.e. 2-1 means first page of problem 2). Also each problem solution should be on a separate set of pages (i.e. not putting parts of two different problems on the same page). At the end of the exam students should scan in their exams in sequence using the cell phone or a scanner (it might be easier to transfer the files to a laptop first) and email them in a file or files (ideally pdf) to philip.mannheim@uconn.edu and juha.javanainen@uconn.edu no later than 15 minutes after the end time of the exam, and the files will be checked to see that they are readable or if a resend is required. Label both the email header and the file or files with your name and the name of the exam. In the email state which problems you have attempted and state how many pages there are for each of the problems.

During the exam students must keep the webex link live on both the computer and the cell phone, but only need to keep the cell phone camera on. Questions that arise should be asked through the chat on the computer webex, and students should arrange for at least the chat portion of the computer webex to be visible to them during the exam. Students can work on the same desk as they place their computer so that their hands are visible. The cell phone should be mounted (scotch tape on a hard vertical surface should suffice) so that the phone shows the computer screen and the entire work area. Proctors will monitor the students through the cell phone camera webex.

1. Let us write the first law for a magnetic material as $dU = T\,dS + H\,dM$, where H is the applied magnetic field and M is the total magnetic moment. In practice a normal metal does not magnetize, while a type I superconductor expels the magnetic field completely, $B = \mu_0 H + M/V = 0$, until the metal turns normal at the critical field $H_c(T)$.

   (a) Calculate the difference of Gibbs free energy between the superconductor and the normal metal as a function of H and T.

2. Consider a non-interacting one-dimensional Fermi gas of atoms with mass m and spin degeneracy g at zero temperature.

   (a) What is the Fermi energy?

   (b) What is the average energy per atom?

3. (a) In a possible formulation of statistical mechanics one postulates that equilibrium density operator ρ is the one that minimizes the entropy $S = -k\mathrm{Tr}(\rho \ln \rho)$ under the constraints of the ensemble. The usual constraint for the canonical ensemble is that the total energy $E = \langle \hat{H} \rangle$ is fixed. However, if it so happens that the system is (or may be regarded as) translationally invariant, total momentum $\mathbf{P} = \langle \hat{\mathbf{P}} \rangle$ is also a constant of the motion. Show that the density operator of the canonical ensemble, in a generalization of the usual form, is

   $$\rho = \frac{1}{Z} e^{-\beta(\hat{H} - \mathbf{v}\cdot\hat{\mathbf{P}})}$$

   The vector constant $\mathbf{v}$ is obviously the overall flow velocity of the gas.

   (b) Take a near-ideal gas with identical atoms of mass m at the temperature T. Show that in the limit of high temperature or low density, the chemical potential is

   $$\mu = -\frac{m\mathbf{v}^2}{2} + kT \ln(n\lambda^3)\,,$$

   where λ is the usual thermal de Broglie wavelength.

   (c) Consider now a pipe with gas flowing at the speed v connected by a tight channel to a chamber containing the same gas, all of this at the temperature T. Given that density of the gas in the flow is n, what is the density in the chamber?

4. In a gravitational-wave detector a mirror mount with mass $m = 1\,\mathrm{kg}$ is hung on a $0.5\,\mathrm{m}$ wire, which makes a pendulum with oscillation frequency $\omega_0 = 4.4\,\mathrm{s}^{-1}$. Let us call the coordinate of small displacements of the pendulum from equilibrium x. The mount is hung in high vacuum, so that the Q value of the pendulum is 10^6 and the damping constant of pendulum amplitude thus is $\gamma = 2.2\times10^{-6}\mathrm{s}^{-1}$. When a force $F(t)$ drives the pendulum, the corresponding "interaction energy" is $H' = -F(t)x$. On the other hand, the equation of motion for the driven pendulum is

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = F(t)/m\,. \tag{1}$$

(a) Argue from the theory of linear response that a force $F(t) = F_0 e^{-i\omega t}$ elicits the stationary response $x(t) = F_0\,\chi_{xx}(\omega)e^{-i\omega t}$.

(b) By comparing the result of part (a) with the proper stationary solution of Eq. (1), find the response function $\chi_{xx}(\omega)$ of the mirror mount.

(c) Now suppose that we are after a burst of gravitational radiation at the frequency $\omega_g = 2\pi \times 500\,\mathrm{Hz}$, which lasts for about $10\,\mathrm{s}$. The radiation therefore appears in a band of width $\Delta\omega \simeq 10^{-1}\mathrm{s}^{-1}$. At room temperature, what is the amplitude of the thermal fluctuations of the mirror that the experiment must contend with?